IMS

MATHS

BOOK-11

* MATRICES *

Def → A matrix is a rectangular array of numbers (real or Complex). The numbers are called the elements of the matrix or entries of the matrix.

Matrices are represented by the brackets ( ) or [ ].

Def → If a matrix A has m rows and n columns then the matrix is said to be of type or order or size $m \times n$.

Ex:- If $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 5 \end{bmatrix}$ then the order of A is 2×3.

Def → A matrix A is said to be a Square matrix if the number of rows in A is equal to the number of columns in A.

Ex:- $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

Note:- (1) An $n \times n$ square matrix is called a Square matrix of type n.

(2) If A is a Square matrix then the diagonal in A from the first element of the first row to the last element of the last row is called the principal diagonal of A. Ex: $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 1 & 10 \end{bmatrix}$

Def A matrix A is said to be a rectangular matrix if the number of rows in A is not equal to the number of columns in A.

Note: A is called a rectangular matrix if A is not a square matrix.

Ex:- $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

A matrix 'A' is said to be a zero matrix if every element of A is equal to zero.

An $m \times n$ zero matrix is denoted by $O_{m \times n}$ or O.

Ex:- $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Def A matrix 'A' is said to be a row matrix if A contains only one row.

Ex:- $[23 \quad 24]$, $[23 \quad 24 \quad 25]$.

→ **Involutory Matrix** :–

A Square matrix A is said to be an involutory matrix if $A^2 = I$.

Ex: If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $A^2 = I$

If $A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ then $A^2 = I$

If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ then $A^2 = I$

→ **Nilpotent Matrix** :–

A square matrix A is said to be a nilpotent matrix if $\exists$ a +ve integer n such that $A^n = 0$. If n is the least +ve integer such that $A^n = 0$, then 'n' is called the index of the nilpotent matrix A.

Ex: Show that $A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$ is a nilpotent matrix.

Sol^n:– $A^2 = A.A$

$$= \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix} \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 0$$

$\therefore$ A is a nilpotent matrix of index 2.

→ **Trace of Matrix** :

Let $A = [a_{ij}]_{n\times n}$. The sum of the elements of A lying along the principal diagonal is called the trace of A. It is denoted by tr A.

i.e. if $A = [a_{ij}]_{n\times n}$ then

$$\text{tr}A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}.$$

Properties

If A and B are two square matrices of order n and λ be a scalar then

1. $\text{tr}(\lambda A) = \lambda\, \text{tr} A$.
2. $\text{tr}(A+B) = \text{tr}A + \text{tr}B$
3. $\text{tr}(AB) = \text{tr}(BA)$

→ **Transpose of Matrix** :–

Let $A = [a_{ij}]_{m\times n}$. Then the matrix $[a_{ji}]_{n\times m}$ obtained from A by changing its rows into columns and columns into rows is called the transpose of A and is denoted by A^T or A'.

Note:– 1. If A is ~~mxn~~ matrix then A^T is $n\times m$ matrix.

$\therefore (j,i)^{th}$ entry of $A^T = (i,j)^{th}$ entry of A.

Ex: Let $A = \begin{bmatrix} 3 & 2+i \\ -5 & 0 \\ \sqrt{2} & 1 \end{bmatrix}_{3\times 2}$ then

$$A^T = \begin{bmatrix} 3 & -5 & \sqrt{2} \\ 2+i & 0 & 1 \end{bmatrix}_{2\times 3}$$

Some important results are given below:—

1. $(A^T)^T = A$ and $(-A)^T = -A^T$
2. $(A+B)^T = A^T + B^T$
3. $(A-B)^T = A^T - B^T$
4. $(KA)^T = KA^T$ where K is any scalar.
5. $(AB)^T = B^T A^T$

→ Symmetric Matrix :—

A square matrix $A = [a_{ij}]$ is said to be symmetric if $A^T = A$.

i.e. $[a_{ji}] = [a_{ij}]$.

Ex: (1) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = A$.

(2). Let $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ then $A^T = A$.

→ Skew Symmetric (Anti Symmetric) Matrix :—

A square matrix $A = [a_{ij}]$ is said to be skew symmetric if $A^T = -A$

i.e. $[a_{ji}] = [-a_{ij}]$.

Ex: Let $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ then

$\therefore A^T = -A$.

Note: (1) the diagonal elements of a skew symmetric matrix must zero.

(2) Below the diagonal and above the diagonal elements of a symmetric matrix must be equal.

(3). A zero matrix is both symmetric as well as skew-symmetric.

* Some Properties of Symmetric & skew symmetric matrices :

→ If A is symmetric matrix then KA is also symmetric matrix.

Sol :— Since A is symmetric matrix.

$\therefore A^T = A$ ——— (1)

Now $(KA)^T = KA^T$

$= KA$ (from (1))

$\therefore$ KA is also Symmetric Matrix.

Similarly If A is Skew Symmetric matrix then KA is also skew-symmetric matrix.

→ If A, B are symmetric then A+B is also symmetric.

Sol'n: Since A, B are symmetric

$\therefore A^T = A$ & $B^T = B$ ——— ①

Now $(A+B)^T = A^T + B^T$

$= A + B$ (from ①)

$\therefore$ A+B is Symmetric matrix.

→ If A, B are Skew symmetric then A+B is also Skew symmetric

Sol'n: Since A & B are skew symmetric

$\therefore A^T = -A$ & $B^T = -B$ ——— ①

Now $(A+B)^T = A^T + B^T$

$= -A - B$

$= -(A+B)$

$\therefore$ A+B is a Skew Symmetric.

→ If A and B are symmetric matrices, show that AB+BA is symmetric and AB−BA is skew-symmetric.

Sol'n: Since A & B are Symmetric

$\therefore A^T = A$ & $B^T = B$.

Now we have

$(AB+BA)^T = (AB)^T + (BA)^T$ $(\because (A+B)^T = A^T + B^T)$

$= B^T A^T + A^T B^T$ $(\because (AB)^T = B^T A^T)$

$= BA + AB$

$= AB + BA$ $(\because A+B = B+A)$

$\therefore$ AB+BA is Symmetric.

Similarly AB−BA is also skew-symmetric.

Prop: If A and B are symmetric matrices, then show that AB is symmetric if and only if A and B Commute i.e. AB = BA.

Proof: Given that A & B are symmetric.

$\therefore A^T = A$ & $B^T = B$ ——— ①

Now suppose that AB = BA

To prove that AB is symmetric.

we have

$(AB)^T = B^T A^T$

$= BA$

$= AB$ $(\because AB = BA)$

$\therefore$ AB is symmetric.

Now Conversely suppose that AB is a Symmetric matrix.

To Prove that AB = BA

we have $AB = (AB)^T$ (∵ AB is symmetric)

$= B^T A^T$

$= BA$ $(\because B^T = A, A^T = A)$

→ If A be any matrix, then Prove that AA^T and A^TA are both Symmetric matrices.

Sol^n :- Let A be any matrix.

we have

$$(AA^T)^T = (A^T)^T \cdot A^T$$
$$= AA^T$$

$\therefore AA^T$ is Symmetric.

we have $(A^TA)^T = (A^T)(A^T)^T$

$$= A^TA$$

$\therefore A^TA$ is Symmetric.

Imp → Show that the matrix B^TAB is Symmetric (or) skew symmetric according as A is Symmetric (or) skew symmetric

Sol^n :- Case(i)

Let A be a symmetric matrix

then $A^T = A$ —— ①

Now we have

$$(B^TAB)^T = B^TA^T(B^T)^T \quad (\because (A^T)^T = A)$$

$$= B^TAB \quad (\text{by } ①)$$

$\therefore B^TAB$ is symmetric.

Case(ii) :- Let A be a skew-Symmetric matrix.

Then $A^T = -A$

Now we have

$$(B^TAB)^T = B^TA^T(B^T)^T$$

$$= B^T(-A)B \quad (\because A^T = -A)$$

$$= -(B^TAB).$$

$\therefore B^TAB$ is skew-symmetric.

Imp. Show that every square matrix is uniquely expressible as the sum of symmetric and skew-Symmetric matrices.

Sol^n Let A be any square matrix.

Then $A = \frac{1}{2}A + \frac{1}{2}A$

$$= \tfrac{1}{2}A + \tfrac{1}{2}A + \tfrac{1}{2}A^T - \tfrac{1}{2}A^T$$

$$= \tfrac{1}{2}(A + A^T) + \tfrac{1}{2}(A - A^T) \quad \text{—— ①}$$

$$= (P + Q) \text{ (say)}$$

where $P = \frac{1}{2}(A + A^T)$ & $Q = \frac{1}{2}(A - A^T)$

Since $P^T = \left[\frac{1}{2}(A + A^T)\right]^T$

$$= \tfrac{1}{2}(A^T + (A^T)^T)$$

$$= \tfrac{1}{2}(A^T + A)$$

$$= \tfrac{1}{2}(A + A^T) \quad (\because A + B = B + A)$$

$$= P$$

$\therefore P$ is symmetric.

Since $Q^T = \left[\frac{1}{2}(A - A^T)\right]^T$

$$= \tfrac{1}{2}(A^T - (A^T)^T)$$

$$= -\tfrac{1}{2}(A - A^T)$$

$$= -Q$$

$\therefore Q$ is skew-symmetrix.

$\therefore A$ ~~is~~ the sum of a symmetric matrix P and skew-symmetric

matrix Q i.e. $P+Q$. ——— (1)

Now to prove uniqueness i.e. we want to prove that the representation (1) of A is unique.

If possible let $A = R+S$ be another representation, where R is symmetric & S is skew-symmetric.

Since R is symmetric & S is skew-symmetric.

$\therefore R^T = R$, $S^T = -S$.

Now $A^T = (R+S)^T = R^T + S^T$

$= R-S$ ——— (2)

and also $A = R+S$ ——— (3)

from (2) & (3) we have

$\therefore A+A^T = R+S+R-S$

$= 2R$

and $A-A^T = R+S-(R-S)$

$= 2S$

$\Rightarrow R = \frac{1}{2}(A+A^T) = P$ &

$S = \frac{1}{2}(A-A^T) = Q$.

$\Rightarrow$ The representation (1) of A as the sum of a symmetric and skew-symmetric matrix is unique.

$\rightarrow$ Show that all positive integral powers of a symmetric matrix is symmetric.

Sol^n:- Let A be a symmetric matrix of order n, then $A^T = A$ ——— (1)

Now $A^m = A \cdot A \cdot A \cdots A$ upto m times where m is a +ve integer.

Now, $(A^m)^T = (A \cdot A \cdots A \text{ upto } m \text{ times})^T$

$= (A^T A^T \cdots A^T \text{ upto } m \text{ times})$

$= (A \cdot A \cdots A \text{ upto } m \text{ times})$ (By (1))

$= A^m$

$\therefore A^m$ is also a symmetric matrix.

Imp $\rightarrow$ Show that +ve odd integral powers of a skew-symmetric matrix are skew symmetric while +ve even integral powers are symmetric.

Sol^n:- Let A be a skew symmetric then $A^T = -A$.

Now let 'm' be a +ve integer we have

$A^m = (A \cdot A \cdot A \cdots A \text{ upto } m \text{ times})$

Now $(A^m)^T = (A \cdot A \cdots A \text{ upto } m \text{ times})^T$

$= A^T \cdot A^T \cdots A^T$ upto m times

$= (-A)(-A)\ldots(-A)$ upto m times $(\because A^T = A)$

$= (-1)^m A^m$

$= -A^m$ or A^m

according as m is odd or even.

$\therefore$ If m is odd +ve integer then $(A^m)^T = -A^m$.

$\therefore$ A^m is skew-symmetric.

If m is an even +ve integer then $(A^m)^T = A^m$.

$\therefore$ A^m is symmetric.

Imp → If U and V are two symmetric matrices, show that UVU is also symmetric. Is UV symmetric always? Explain and illustrate by an example.

Sol'n :- Since U & V are symmetric matrices.

$\therefore$ $U^T = U$ & $V^T = V$.

Now we have $(UVU)^T = U^T V^T U^T$

$= UVU$

$\therefore$ UVU is symmetric.

Since U & V are symmetric

$\therefore$ UV is symmetric iff $UV = VU$.

If $UV \neq VU$ then UV is not symmetric.

Ex: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

then $AB = \begin{bmatrix} 8 & 11 \\ 13 & 18 \end{bmatrix}$ &

$BA = \begin{bmatrix} 8 & 13 \\ 11 & 18 \end{bmatrix}$.

$\therefore$ $AB \neq BA$.

Now $(AB)^T = B^T A^T$

$= BA$

$\neq AB$ $(\because AB \neq BA)$

$\therefore$ AB is not symmetric.

→ Let A be a square matrix, Prove that

(i) $A + A^T$ is a symmetric matrix o

(ii) $A - A^T$ is a skew symmetric mat

Sol'n :- (i) we have

$(A + A^T)^T = A^T + (A^T)^T$ $(\because (A+B)^T = A^T + B^T)$

$= A^T + A$ $(\because (A^T)^T)$

$= A + A^T$ $(\because A + B = B + A)$

$\therefore$ $A + A^T$ is symmetric.

(ii) $(A - A^T)^T = A^T - (A^T)^T$ $(\because (A-B)^T = A^T - B^T)$

$= A^T - A$ $(\because (A^T)^T = A)$

$= -(A - A^T)$

$\therefore$ $A - A^T$ is skew-symmetric

* Conjugate of Matrix :-

Definition :- The matrix obtained from any given matrix A on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\bar{A}$.

i.e. If $A = [a_{ij}]_{m\times n}$ then

$$\bar{A} = [\bar{a}_{ij}]_{m\times n}.$$

Ex: Let $A = \begin{bmatrix} 1 & 1+i \\ 3 & 1-i \end{bmatrix}$ then

$$\bar{A} = \begin{bmatrix} 1 & 1-i \\ 3 & 1+i \end{bmatrix}$$

Note: If all the elements of A are purely real then $A = \bar{A}$.

→ **Properties of Conjugate** :-

Let $\bar{A}$ and $\bar{B}$ be the conjugates of A and B then

(i) $(\bar{\bar{A}}) = A$ (ii) $(\overline{A+B}) = \bar{A} + \bar{B}$

(iii) $(\overline{KA}) = \bar{K}\bar{A}$ where K is any complex number.

(iv) $(\overline{AB}) = \bar{A}\bar{B}$, A & B are conformable for multiplication.

* Tranjugate (or) Transposed Conjugate of matrix :-

The transpose of the conjugate of a matrix A is called transpose conjugate of A and ~~is denoted by~~ A^{θ} or A^{K}.

i.e. $A^{\theta} = (\bar{A})^{T}$

Ex:- $A = \begin{bmatrix} 2 & 1+i & 0 \\ 3 & 2 & i \end{bmatrix}$ then $A^{\theta} = \begin{bmatrix} 2 & \\ 1-i & \\ 0 & \end{bmatrix}$

Note: It is also possible that $(\bar{A})' = (\overline{A'})$.

→ **Properties of transpose Conjugate**

Let A^{θ} and B^{θ} be the transpose conjugates of A and B then

(i) $(A^{\theta})^{\theta} = A$

(ii) $(A+B)^{\theta} = A^{\theta} + B^{\theta}$, A & B are of the same order.

(iii) $(KA)^{\theta} = \bar{K}A^{\theta}$, where K is any complex number

(iv) $(AB)^{\theta} = B^{\theta}A^{\theta}$, A & B are conformable to multiplication

* Hermitian Matrix :-

A square matrix A is said to be a Hermitian matrix if the transpose of the conjugate matrix is equal to the matrix itself. i.e. $A^{\theta} = A$.

Note: The elements on the principal diagonal must be all real numbers. i.e. $\bar{a}_{ii} = a_{ii}$.

Ex: $\begin{bmatrix} 2 & 3+2i \\ 3-2i & 7 \end{bmatrix}$, $\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$ and

$$\begin{bmatrix} 2 & 1+i & -2+3i \\ 1-i & 1 & 5-6i \\ -2-3i & 5+6i & 0 \end{bmatrix}$$ are

Hermitian matrices.

→ **Skew – Hermitian Matrix:–**

A square matrix A is said to be skew-Hermitian if $A^\theta = -A$.

Note: The elements on the principal diagonal must be purely imaginary number or zero.

Ex: $\begin{bmatrix} 2i & 5+4i \\ -5+4i & 0 \end{bmatrix}$, $\begin{bmatrix} 4i & 2+i & 3 \\ -2+i & 0 & 4i \\ -3 & 4i & -3i \end{bmatrix}$

are skew-Hermitian matrices.

→ **Some Properties of Hermitian & Skew-Hermitian matrices:–**

→ If A is a Hermitian matrix, show that iA is skew Hermitian.

Sol^n:– Since A is Hermitian

$\therefore A^\theta = A$.

we have $(iA)^\theta = \bar{i}A^\theta \; (\because (kA)^\theta = \bar{k}A^\theta)$

$= (-i)A^\theta$

$= -(iA^\theta)$

$= -(iA) \; (\because A^\theta = A)$

$\therefore (iA)^\theta = -(iA)$

$\therefore$ iA is skew Hermitian.

→ If A is a skew-Hermitian matrix, then show that iA is Hermitian.

Sol^n:– Since A is skew-Hermitian.

$\therefore A^\theta = -A$

we have $(iA)^\theta = \bar{i}A^\theta$

$= -i(-A) \; (\because A^\theta = -A)$

$= iA$

$\therefore (iA)^\theta = iA$

$\therefore$ iA is Hermitian.

→ If A, B are Hermitian or skew Hermitian then $A+B$ is also Hermitian or skew-Hermitian.

Sol^n:– (i) Given that A & B are Hermitian.

$\therefore A^\theta = A$, $B^\theta = B$

we have $(A+B)^\theta = A^\theta + B^\theta$

$= (A+B)$

$\therefore$ $A+B$ is Hermitian.

(ii) Similarly it can be easily done.

→ If A and B are two $n\times n$ matrices then show that

(i) $(-A)' = -(A')$ (ii) $(-A)^\theta = -(A^\theta)$

(iii) $(A-B)' = A' - B'$ (iv) $(A-B)^\theta = A^\theta - B^\theta$

Sol^n :- we have $(-A)' = \{(-1)A\}'$

$= (-1)A'$

$= -A'$

(ii) $(-A)^\theta = \{(-1)A\}^\theta = (\overline{-1})A^\theta$

$= -A^\theta$ $(\because (\overline{-1}) = -1)$

(iii) we have

$(A-B)' = \{A + (-B)\}' = A' + (-B)'$

$= A' - B^T$

(iv) $(A-B)^\theta = \{A + (-B)\}^\theta$

$= A^\theta + (-B)^\theta$

$= A^\theta - B^\theta$

→ If A & B are Hermitian then show that $AB+BA$ is Hermitian and $AB-BA$ is skew Hermitian.

Sol^n :- Given that A & B are Hermitian.

$\therefore A^\theta = A$ & $B^\theta = B$ —— (1)

(i) Now we have $(AB+BA)^\theta = (AB)^\theta + (BA)^\theta$

$= B^\theta A^\theta + A^\theta B^\theta$

$= BA + AB$

$= AB + BA$

$\therefore AB+BA$ is Hermitian.

(ii) Now we have $(AB-BA)^\theta = (AB)^\theta - (BA)^\theta$

$= B^\theta A^\theta - A^\theta B^\theta$

$= BA - AB$

$= -(AB - BA)$

$\therefore AB - BA$ is Hermitian.

→ If A be any square matrix then prove that $A + A^\theta$, AA^θ, $A^\theta A$ are all Hermitian and $A - A^\theta$ is skew Hermitian.

Sol^n :- Given that A is any square matrix.

(i) we have $(A + A^\theta)^\theta = A^\theta + (A^\theta)^\theta$

$= A^\theta + A$

$= A + A^\theta$

$\therefore A + A^\theta$ is Hermitian.

(ii) we have $(AA^\theta)^\theta = (A^\theta)^\theta A^\theta$

$= AA^\theta$

$\therefore AA^\theta$ is Hermitian.

(iii) we have $(A^\theta A)^\theta = A^\theta (A^\theta)^\theta$

$= A^\theta A$

$\therefore A^\theta A$ is Hermitian.

(iv) $(A - A^\theta)^\theta = A^\theta - (A^\theta)^\theta$

$= A^\theta - A$

$= -(A - A^\theta)$

$\therefore A - A^\theta$ is skew - Hermitian.

Imp → Show that the matrix $B^{\theta}AB$ is Hermitian or skew Hermitian according as A is Hermitian (or) skew-Hermitian.

Sol'n :- Case (i)

Since A is Hermitian

$\therefore A^{\theta} = A$.

Now we have $(B^{\theta}AB)^{\theta} = B^{\theta}A^{\theta}(B^{\theta})^{\theta}$

$= B^{\theta}AB$

$\therefore B^{\theta}AB$ is Hermitian.

Case (ii) :- Since A is skew-Hermitian

$\therefore A^{\theta} = -A$

we have $(B^{\theta}AB)^{\theta} = B^{\theta}A^{\theta}(B^{\theta})^{\theta}$

$= B^{\theta}(-A)B$

$= -(B^{\theta}AB)$

$\therefore B^{\theta}AB$ is skew-Hermitian.

Imp → Prove that every square matrix A is uniquely expressed as the sum of a Hermitian and a skew-Hermitian matrix.

Sol'n :- Since A is any square matrix.

$\therefore A + A^{\theta}$ is Hermitian and $A - A^{\theta}$ is skew-Hermitian.

$\therefore \frac{1}{2}(A + A^{\theta})$ is Hermitian &

$\frac{1}{2}(A - A^{\theta})$ is skew-Hermitian.

Now we have

$A = \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta})$

$= P + Q$ (say) —— (1)

where P is Hermitian & Q is skew-Hermitian.

Now to prove uniqueness.

i.e. we want to prove that the representation (1) of A is unique.

If possible let $A = R + S$ be another representation, where R is Hermitian & S is skew-Hermitian.

Now since R is Hermitian & S is skew-Hermitian.

$\therefore R^{\theta} = R$, $S^{\theta} = -S$.

Now $A^{\theta} = (R+S)^{\theta}$

$= R^{\theta} + S^{\theta}$

$= R - S$ and also —— (2)

$A = R + S$ —— (3)

from (2) & (3) we have

$\therefore A + A^{\theta} = R + S + R - S$

$= 2R$

and $A - A^{\theta} = R + S - (R - S)$

$= 2S$

$\Rightarrow R = \frac{1}{2}(A + A^{\theta}) = P$ and

$S = \frac{1}{2}(A - A^{\theta}) = Q$

$\Rightarrow$ The representation on (1) of A as the sum of a symmetric and skew-symmetric matrix is unique.

Imp → Prove that $\bar{A}$ is Hermitian (or) skew-Hermitian according as A is Hermitian or skew-Hermitian.

Sol'n :- Case (i) :- Given that A is Hermitian.

$\therefore A^{\theta} = A$

To Prove $\bar{A}$ is Hermitian.

Now we have

$$(\bar{A})^{\theta} = \left[\overline{(\bar{A})}\right]'$$

$$= A' \quad (\because \overline{(\bar{A})} = A)$$

$$= (A^{\theta})' \quad (\because A \text{ is Hermitian} \Rightarrow A = A^{\theta}).$$

$$= \left((\bar{A})'\right)' \quad (\because A^{\theta} = (\bar{A})')$$

$$= \bar{A}$$

$\therefore \bar{A}$ is Hermitian.

(ii) Given that A is Skew-Hermitian.

$\therefore A^{\theta} = -A$.

we have $(\bar{A})^{\theta} = \left[\overline{(\bar{A})}\right]'$

$$= A'$$

$$= (-A^{\theta})' \quad (\because A^{\theta} = -A)$$

$$= -\left[(\bar{A})'\right]'$$

$$= -\bar{A}.$$

$\therefore \bar{A}$ is also Skew-Hermitian.

Imp. Show that every square matrix A can be uniquely expressed as $P+iQ$, where P & Q are Hermitian matrices.

Sol'n :- Let $P = \frac{1}{2}(A + A^{\theta})$ and

$Q = \frac{1}{2i}(A - A^{\theta})$

Then $P + iQ = \frac{1}{2}(A + A^{\theta}) + i\frac{1}{2i}(A - A^{\theta})$

$$= \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta})$$

$$= \frac{1}{2}A + \frac{1}{2}A$$

$$= A$$

$\therefore P + iQ = A$ (or) $A = P + iQ$ — (

Now we prove that P & Q are Hermitian.

$$P^{\theta} = \left(\frac{1}{2}(A + A^{\theta})\right)^{\theta} = \frac{1}{2}(A + A^{\theta})$$

$$= \frac{1}{2}(A^{\theta} + A)$$

$$= \frac{1}{2}(A + A^{\theta})$$

$\therefore P$ is Hermitian.

Now $Q^{\theta} = \left[\frac{1}{2i}(A - A^{\theta})\right]^{\theta}$

$$= \left(\overline{\frac{1}{2i}}\right)(A - A^{\theta})^{\theta}$$

$$= \frac{1}{-2i}(A^{\theta} - A)$$

$$= \frac{-1}{2i}\left[-(A - A^{\theta})\right]$$

$$= \frac{1}{2i}(A - A^{\theta})$$

$\therefore Q^{\theta} = Q$

$\therefore Q$ is Hermitian.

$\therefore$ we have expressed A in the form $P+iQ$.

where P & Q are Hermitian.

Now we prove that the expression ① is unique.

Let us suppose that $A = R + iS$ where R and S are Hermitian.

Since R & S are Hermitian

$\therefore R^{\theta} = R$ & $S^{\theta} = S$.

Now $A^{\theta} = (R+iS)^{\theta}$

$= R^{\theta} + (iS)^{\theta}$

$= R + \bar{i}S^{\theta} = R - iS$ — ②

Also $A = R + iS$ — ③

from ② & ③ we have

$A + A^{\theta} = (R+iS) + (R-iS)$

$= 2R$

and $A - A^{\theta} = (R+iS) - (R-iS)$

$= 2iS$

$\Rightarrow \frac{1}{2}(A+A^{\theta}) = R$ and $\frac{1}{2i}(A-A^{\theta}) = S$

$\Rightarrow R = P$ and $S = Q$.

$\therefore$ The expression ① for A is unique.

→ If $AB = A$ and $BA = B$ then $B'A' = A'$ and $A'B' = B'$ and hence Prove that A' and B' are idempotent.

Sol^n :- Since $AB = A \Rightarrow (AB)' = A'$

$\Rightarrow B'A' = A'$

and since $BA = B \Rightarrow (BA)' = B'$

$\Rightarrow A'B' = B'$

Now we prove that A' is idempotent.

~~we have~~ $(A')^2 = A' \cdot A'$

$= A'(B'A')$

$= (A'B')A'$

$= B'A'$ $(\because A'B' = B')$

$= A'$

$\therefore A'$ is Idempotent.

Now we prove that B' is idempotent

we have $(B')^2 = B' \cdot B'$

$= B'(A'B')$

$= (B'A')B'$

$= A'B'$ $(\because B'A' = A')$

$= B'$ $(\because A'B' = B')$

$\therefore B'$ is idempotent.

Determinants

Definition:-

To every square matrix, we associate a unique number called the determinant of matrix.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ - & - & - & \cdots & - \\ - & - & - & \cdots & - \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$

is any square matrix of order n then the determinant of A is denoted by $|A|$ or $\det A$ or Δ.

i.e. $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ - & - & - & \cdots & - \\ - & - & - & \cdots & - \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$

The numbers $a_{11}, a_{12}, \ldots$ are the elements of the determinant.

Note: (1) In determinant, number of rows must be equal to number of columns.

(2) The determinants has a value.

(3) Determinants are used for judging the invertibility of square matrices.

(4) Determinants are also used to solve the system of linear equations.

Ex:- $|7|$, $\begin{vmatrix} 3 & 5 \\ 7 & 9 \end{vmatrix}$, $\begin{vmatrix} 1 & 7 & -5 \\ 3 & -4 & 8 \\ 0 & 9 & 2 \end{vmatrix}$ are the determinants of order 1, 2, 3 respectively.

→ The determinant of 1×1 matrix $[a]$ is defined to be a.

→ **Determinant of order 2:**

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the number $ad-bc$ is called the determinant of A. It is denoted by $|A|$.

i.e. $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$.

*Minors and Cofactors :-

→**Minor** :- If a_{ij} is an element which is in the i^{th} row and j^{th} column of a square matrix A, the determinant of the matrix obtained by deleting the i^{th} row and j^{th} column of A is called minor of A. It is denoted by M_{ij}.

Note:- If $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then

M_{11} = minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$= a_{22}a_{33} - a_{32}a_{23}$.

M_{12} = Minor of $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$= a_{21}a_{33} - a_{31}a_{23}$.

Cofactor :- If a_{ij} is an element which is in the i^{th} row and j^{th} Column of a square matrix A, then the product of $(-1)^{i+j}$ and the minor of a_{ij} is called Cofactor of a_{ij}. It is denoted by A_{ij}.

Note: If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

A_{11} = Cofactor of $a_{11} = (-1)^{1+1} M_{11}$

$$= +1 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$= a_{22}a_{33} - a_{32}a_{23}$$

A_{12} = Cofactor of $a_{12} = (-1)^{1+2} M_{12}$

$$= -1 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$= -(a_{21}a_{33} - a_{31}a_{23})$$

And so on.

→ If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then

$$\Delta = \sum_{j=1}^{3} (-1)^{i+j} a_{ij} M_{ij} = \sum_{j=1}^{3} a_{ij} A_{ij}$$

$$= a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3}$$

for either $i=1$ or $i=2$ or $i=3$

i.e. $\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$,

$\Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$, (Row)

$\Delta = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$.

(or)

$$\Delta = \sum_{i=1}^{3} (-1)^{i+j} a_{ij} M_{ij} = \sum_{i=1}^{3} a_{ij} A_{ij}$$

$$= a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j}$$

for either $j=1$ or $j=2$ or $j=3$.

i.e. $\Delta = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$,

$\Delta = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$

$\Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$

(Column)

Note:– The determinant of a square matrix A is equal to the sum of the products of the elements of a row (or Column) of A with their corresponding cofactors.

Problem

→ Find the value of the determinant of the matrix.

$$A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$$

Sol'n:– we have $|A| = \begin{vmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{vmatrix}$

$$= a \begin{vmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{vmatrix}$$

$$= ab \begin{vmatrix} c & 0 \\ 0 & d \end{vmatrix}$$

$$= ab(cd - 0) = abcd$$

Note: (1). The value of the determinant of diagonal matrix is equal to the product of the elements lying along its principal diagonal.

(2) Let I_n be a unit matrix of order 'n' then $|I_n| = 1$.

$\therefore$ The value of the determinant of a unit matrix is always equal to 1.

→ Find the value of the determinant of the matrix $A = \begin{bmatrix} a & h & g & f \\ 0 & b & c & e \\ 0 & 0 & d & k \\ 0 & 0 & 0 & l \end{bmatrix}$

Sol'n: We have

$$|A| = \begin{vmatrix} a & h & g & f \\ 0 & b & c & e \\ 0 & 0 & d & k \\ 0 & 0 & 0 & l \end{vmatrix}$$

$$= a \begin{vmatrix} b & c & e \\ 0 & d & k \\ 0 & 0 & l \end{vmatrix}$$

$$= ab \begin{vmatrix} d & k \\ 0 & l \end{vmatrix}$$

$$= ab(dl - 0) = abdl.$$

Note: (1) The value of the determinant of an upper triangular matrix (i.e. in which all the elements below the principal diagonal are zero) is equal to the product of the elements along the principal diagonal.

(2) The value of the determinant of a lower triangular matrix is equal to the product of the elements along the principal diagon[al].

* Properties of Determinants:-

→ The value of a determinant does not change when rows and columns are interchanged.

i.e. $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & \cdots & - \\ - & - & \cdots & - \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ + & + & \cdots & + \\ + & + & \cdots & + \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix}$

→ If A be an n-rowed square matri[x] then $|A| = |A^T|$.

→ If any two rows (or two colum[ns]) of a determinant are interchanged, then the value of the determinant is multiplied by −1.

→ If all the elements of one ro[w] (or one column) of a determinant are multiplied by the same numbe[r] K, then the value of the new determinant is K times the value of the given determinant.

i.e. $\begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

→ If A be an n-rowed square matrix, and K be any scalar then $|KA| = K^n|A|$.

i.e. $|KA| = \begin{vmatrix} Ka_{11} & Ka_{12} & \cdots & Ka_{1n} \\ Ka_{21} & Ka_{22} & \cdots & Ka_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ Ka_{n1} & Ka_{n2} & \cdots & Ka_{nn} \end{vmatrix}$

$= K.K \cdots K(n \text{ times}) \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$

$= K^n|A|$.

→ If two rows (or columns) of a determinant are identical, then the value of the determinant is zero.

i.e. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0$

→ In a determinant, the sum of the product of the elements of any row (column) with the cofactors of the corresponding elements of any other row (column) is zero.

Ex:- $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 6 & 7 & 8 \end{vmatrix} = 1(40-28) - 2(-24)+3(-30)$

Now $1(16-21) - 2(8-18) + 3(7-12)$

$= -5+20-15 = 0$

Note:- Let

$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ be a determinant of order 3.

Let A_1, B_1, C_1 etc be the cofactors of the elements a_1, b_1, c_1 etc in Δ.

Then we have

$a_1A_1 + b_1B_1 + c_1C_1 = \Delta$

$a_1A_2 + b_1B_2 + c_1C_2 = 0$

$a_1A_3 + b_1B_3 + c_1C_3 = 0$

$a_2A_2 + b_2B_2 + c_2C_2 = \Delta$

$a_2A_1 + b_2B_1 + c_2C_1 = 0$ etc.

→ Let A be a square matrix of order 'n' then show that

(i) $|\bar{A}| = \overline{|A|}$ (ii) $|A^\theta| = \overline{|A|}$

Sol'n :- Let $A = [a_{ij}]_{n\times n}$ then we have $\bar{A} = [\overline{a_{ij}}]_{n\times n}$

we have $|\bar{A}| = |\overline{a_{ij}}| = \overline{|a_{ij}|} = \overline{|A|}$

(ii) we have $A^\theta = (\bar{A}')$

$\therefore |A^\theta| = |(\bar{A}')| = \overline{|A'|}$

$= \overline{|A|}$ $(\because |A| = |A'|)$

→ show that the determinant of a Hermitian matrix is always a real number.

Sol'n :- Let A be a Hermitian matrix.

Then $A^\theta = A$

$\therefore |A| = |A^\theta| = \overline{|A|}$

we know that if z is a complex number such that $z = \bar{z}$ then z is real.

$\therefore |A| = |\bar{A}|$

$\Rightarrow |A|$ is a real number.

→ Show that the value of the determinant of a skew-symmetric matrix of odd order is always zero.

Sol'n :- Let $A = \begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$ be a skew-symmetric matrix of order 3.

Then $|A| = \begin{vmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{vmatrix}$

$= (-1)^3 \begin{vmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{vmatrix}$

$= - \begin{vmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{vmatrix}$ [By interchanging the rows & columns]

$= -|A|$

$\therefore |A| = -|A| \Rightarrow 2|A| = 0$

$\Rightarrow |A| = 0$

Note : If A, B are square matrices conformable for multiplication then $|AB| = |A||B|$.

→ If A, B are square matrices each of order 'n' such that $|AB| = 0$ then prove that $|A| = 0$ or $|B| = 0$

Sol'n :- Since $|AB| = 0 \Rightarrow |A||B| = 0$

$\Rightarrow |A| = 0$ or $|B| = 0$.

* Adjoint Matrix :-

The transpose of the matrix obtained by replacing the element of a square matrix A by the corresponding cofactors is called the adjoint matrix of A.

It is denoted by Adj A or adj A.

Note : (1)

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$\text{Adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

(2) If

$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ then $\text{Adj } A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$

$\boxed{\text{Imp}}$ → If A be any n-rowed square matrix then $(\text{adj}A)A = A(\text{adj}A) = |A| I$

Sol'n :-

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ - & - & - & \cdots & - \\ - & - & - & \cdots & - \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$ th

$$\text{adj} A = \begin{bmatrix} A_{11} & A_{21} & A_{31} & \cdots & A_{n1} \\ A_{12} & A_{22} & A_{32} & \cdots & A_{n2} \\ - & - & - & \cdots & - \\ - & - & - & \cdots & - \\ A_{1n} & A_{2n} & A_{3n} & \cdots & A_{nn} \end{bmatrix}$$

Now we have

$$(\text{adj}A)A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ - & - & \cdots & - \\ - & - & \cdots & - \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & \cdots & - \\ - & - & \cdots & - \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The $(i,j)^{th}$ element in the matrix $(\text{adj}A)A$ is

$$A_{1i}a_{1j} + A_{2i}a_{2j} + A_{3i}a_{3j} + \cdots + A_{ni}a_{nj}$$
$$= |A|, \text{ if } i = j$$
$$= 0 \text{ if } i \neq j.$$

$$\Rightarrow (\text{adj}A)A = \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & \cdots & 0 \\ - & - & - & \cdots & - \\ - & - & - & \cdots & - \\ 0 & 0 & 0 & \cdots & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ - & - & \cdots & - \\ 0 & 0 & \cdots & 1 \end{bmatrix} = |A| I_n$$

Similarly $A(\text{adj}A) = |A| I_n$

$\therefore (\text{adj}A)A = A(\text{adj}A) = |A| I_n$.

Note: If $|A| \neq 0$ then

$$A\left(\frac{1}{|A|}(\text{adj}A)\right) = \left(\frac{1}{|A|}(\text{adj}A)\right)A = I.$$

$[\because A(\text{adj}A) = (\text{adj}A)A = |A| I_n]$

* Inverse of a Square Matrix:

Let A be any square matrix, if there exists a square matrix B such that $AB = BA = I$. Then the matrix B is called inverse of A.

Note: (1) For AB, BA to be both defined and equal, it is necessary that A and B are both square matrices of same order.

(2) → A rectangular matrices cannot have inverse.

(3) → Every square matrix cannot have inverse.

→ A matrix is said to be invertible if it has inverse.

Imp → Every Invertible matrix has unique inverse.

Sol^n: Let A be an invertible matrix. Let B and C be two inverses of A.

Then $AB = BA = I$ —— (1)

and $AC = CA = I$ —— (2)

Now we have $B = BI$

$= B(AC)$ (By (2))

$= (BA)C$ (By associative property)

$= IC$ (By (1))

$= C$

$\therefore B = C$

$\therefore$ A has unique inverse.

Note :- (1) If A is an invertible matrix then its inverse is denoted by A^{-1}.

$\therefore AA^{-1} = A^{-1}A = I$.

(2) Since $I \cdot I = I$, we have $I^{-1} = I$. i.e. the inverse of a unit matrix is itself.

(3). Since $A^{-1}A = AA^{-1} = I \Rightarrow (A^{-1})^{-1} = A$.

(4). If A is invertible matrix and if $A = B$ then $A^{-1} = B^{-1}$.

Imp. The necessary and sufficient condition for a square matrix A to possess the inverse is that $|A| \neq 0$.

Proof :-

N.C. : Let A be a square matrix. Let B be the inverse of A.

Then $AB = I$

$\Rightarrow |AB| = |I|$

$\Rightarrow |A||B| = 1$

$\Rightarrow |A| \neq 0$.

S.C :- Let $|A| \neq 0$

To prove the matrix A possess the inverse.

we know that

$A(\text{adj} A) = (\text{adj} A)A = |A| I$.

$\Rightarrow A\left(\frac{1}{|A|}\text{adj} A\right) = \left(\frac{1}{|A|}\text{adj} A\right)A = I$

$(\because |A| \neq 0)$

$\Rightarrow AB = BA = I$ where

$B = \frac{1}{|A|}\text{adj} A$.

$\Rightarrow B$ is the inverse of A.

$\Rightarrow A$ has the inverse.

Note: If $|A| \neq 0$ then $A^{-1} = \frac{1}{|A|}(\text{adj} A)$

* Singular and Non-Singular Matrices :-

$\rightarrow$ A square matrix 'A' is said to be singular if $|A| = 0$.

$\rightarrow$ A square matrix A is said to be Non-singular if $|A| \neq 0$.

$\rightarrow$ Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}.$$ Also verify your resu

Sol'n :- We have $|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix}$

$= 0(-1) - 1(-8) + 2($

$= 8 - 10$

$= -2 \neq 0$.

$\therefore A^{-1}$ exists.

Now the co-factor matrix of

$$A = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix} = B \text{ (say)}$$

Now $\text{adj} A = B^T$

$$= \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

Now $A^{-1} = \dfrac{\text{adj} A}{|A|}$

$$= -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

Verification:-

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Similarly $A^{-1}A = I_3$

$\therefore A^{-1}A = AA^{-1} = I_3$

→ Find the inverse of the matrix

$$S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$ and show that

SAS^{-1} is a diagonal matrix

where $A = \frac{1}{2}\begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$

Sol'n :- $|S| = 0(-1) - 1(-1) + 1(1)$

$= 1+1 = 2$

$\therefore |S| \neq 0$

$\therefore S^{-1}$ exists.

Cofactors matrix of $S = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$= B$ (say)

Now $\text{adj} A = B^T = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Now $S^{-1} = \dfrac{\text{adj} S}{|S|} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Now $SA = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} 0 & 2a & 2a \\ 2b & 0 & 2b \\ 2c & 2c & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & a \\ b & 0 & b \\ c & c & 0 \end{bmatrix}$$

$\therefore SAS^{-1} = \begin{bmatrix} 0 & a & a \\ b & 0 & b \\ c & c & 0 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} 2a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 2c \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

$\therefore SAS^{-1}$ is a diagonal matrix.

(Imp) If A, B be two n-rowed non-singular matrices, then AB is also non-singular and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof :- Let A & B be two n-rowed non-singular matrices.

$\therefore |A| \neq 0, \ |B| \neq 0$.

Now we have $|AB| = |A|\,|B|$

$\Rightarrow |AB| \neq 0 \quad (\because |A| \neq 0, |B| \neq 0)$

$\therefore AB$ is non-singular.

$\therefore AB$ has inverse.

$\therefore AB$ is invertible.

Let us define a matrix C by the relation $C = B^{-1}A^{-1}$.

Then $C(AB) = (B^{-1}A^{-1})(AB)$

$= B^{-1}(A^{-1}A)B$

$= B^{-1}IB = B^{-1}B = I$

Also $(AB)C = (AB)(B^{-1}A^{-1})$

$= A(BB^{-1})A^{-1}$

$= AIA^{-1}$

$= AA^{-1} = I$

$\therefore C(AB) = (AB)C = I$

$\therefore C = B^{-1}A^{-1}$ is the inverse of AB.

→ If A be an $n \times n$ non-singular matrix then $(A^T)^{-1} = (A^{-1})^T$

Sol^n:- Since $|A| \neq 0$

Now $|A^T| = |A| \neq 0$.

$\therefore A^T$ is non-singular.

We know that $AA^{-1} = A^{-1}A = I$

$\therefore (AA^{-1})^T = (A^{-1}A)^T = I^T$

$\Rightarrow (A^{-1})^T A^T = A^T (A^{-1})^T = I$.

$\therefore (A^{-1})^T$ is the inverse of A^T.

$\therefore (A^T)^{-1} = (A^{-1})^T$.

→ If A be an $n \times n$ non-singular matrix then $(A^{-1})^\theta = (A^\theta)^{-1}$.

Sol^n:- Since A is non-singular mat.

$\therefore |A| \neq 0$. $\quad (\because |A| = |$

Now $|A^\theta| = \overline{|A|} \neq 0$.

$\therefore A^\theta$ is non-singular.

We know that $AA^{-1} = A^{-1}A = I$.

$\Rightarrow (AA^{-1})^\theta = (A^{-1}A)^\theta = I^\theta$

$\Rightarrow (A^{-1})^\theta A^\theta = A^\theta (A^{-1})^\theta = I \quad (\because I^\theta = (\bar{I})' = I' = I)$

$\Rightarrow (A^{-1})^\theta$ is the inverse of A^θ.

$\therefore (A^\theta)^{-1} = (A^{-1})^\theta$

Note :- If A is a square matrix then $\operatorname{adj} A^T = (\operatorname{adj} A)^T$.

→ If A is a symmetric matrix then show that $\operatorname{adj} A$ is symmetric.

Sol^n :- Since A is symmetric.

$\therefore A^T = A$.

Now we have $(\operatorname{adj} A)^T = \operatorname{adj} A^T$

$= \operatorname{adj} A \quad (\because A^T = A)$

$\therefore \operatorname{adj} A$ is symmetric.

→ If the non-singular matrix A is symmetric then A^{-1} is also symmetric.

Sol'n :- Since A is non-singular.

i.e. $|A| \neq 0$.

$\therefore A^{-1}$ exists.

And A is symmetric.

$\therefore A^T = A$.

Now we have $(A^{-1})^T = (A^T)^{-1}$

$= A^{-1}$ $(\because A^T = A)$

$\therefore A^{-1}$ is symmetric.

→ Show that if A is a non-singular matrix, then $\det(A^{-1}) = (\det A)^{-1}$

Sol'n :- Since A is non-singular i.e. $|A| \neq 0$.

$\therefore A^{-1}$ exists.

$\therefore A^{-1}A = AA^{-1} = I$

Now we have

$A^{-1}A = I \Rightarrow |A^{-1}A| = |I|$

$\Rightarrow |A^{-1}||A| = 1$

$\Rightarrow |A^{-1}| = \frac{1}{|A|}$

$\Rightarrow |A^{-1}| = |A|^{-1}$

$\Rightarrow \det(A^{-1}) = (\det A)^{-1}$.

→ If B is non-singular, prove that the matrices A & $B^{-1}AB$ have the same determinant, A and B being both square matrices.

Sol'n :- Since B is non-singular.

$\therefore |B| \neq 0$.

Now we have $|B^{-1}AB| = |B^{-1}||A||B|$

$= |B^{-1}||B||A|$

$= |B^{-1}B||A|$

$= |I||A|$

$= 1 \cdot |A|$

$\therefore |B^{-1}AB| = |A|$

→ If A be an $n \times n$ square matrix then prove that $|\text{adj} A| = |A|^{n-1}$.

Sol'n :- we have $A(\text{adj} A) = |A| \cdot I_n$

$\Rightarrow |A(\text{adj} A)| = ||A| \cdot I_n|$

$\Rightarrow |A||\text{adj} A| = |A|^n |I_n|$

$\Rightarrow |A||\text{adj} A| = |A|^n \cdot 1$ $(\because |kA| = k^n|A|$ & $|I_n| = 1)$

$\Rightarrow |\text{adj} A| = |A|^{n-1}$ if $|A| \neq 0$.

Note :- (1). From the above example if $|A| = 0$ then $|\text{adj} A| = 0$.

(2) From the above example if $|A| \neq 0$ then $|\text{adj} A| \neq 0$.

i.e. if A is non-singular then adj$A$ is non-singular.

→ If A is a non-singular matrix then show that

$\text{adj}(\text{adj} A) = |A|^{n-2} A$.

Sol'n :- Since A is non-singular.

$\therefore |A| \neq 0$ and A^{-1} exists.

we know that $A(\text{adj} A) = |A| I_n$ —— (1)

Taking adj A in place of A in (1), we get

$(\text{adj}A)(\text{adj}(\text{adj}A)) = |\text{adj}A| I_n$

$\Rightarrow (\text{adj}A)(\text{adj}(\text{adj}A)) = |A|^{n-1} I_n$

$(\because |\text{adj}A| = |A|^{n-1})$

Pre-multiplying bothsides by A, we get

$(A(\text{adj}A))(\text{adj}(\text{adj}A)) = |A|^{n-1}(A I_n)$

$\Rightarrow (|A| I_n)[\text{adj}(\text{adj}A)] = |A|^{n-1} A$

$(\because A I_n = A)$

$\Rightarrow |A|[I_n \text{adj}(\text{adj}A)] = |A|^{n-1} A$.

$\Rightarrow |A|[\text{adj}(\text{adj}A)] = |A|^{n-1} A$

Since $|A| \neq 0$

$\therefore \boxed{\text{adj}(\text{adj}A) = |A|^{n-2} A}$

→ If A & B are two non-singular matrices of the same type then $(\text{adj}AB) = (\text{adj}B)(\text{adj}A)$.

Sol'n :- Since $|A| \neq 0$, $|B| \neq 0$

$\therefore |AB| \neq 0$.

we know that $(AB)(\text{adj}(AB)) = |AB| I$

$= (\text{adj}(AB))(AB)$ (1)

Now $(AB)(\text{adj}B\,\text{adj}A) = A(B\,\text{adj}B)\text{adj}A$

(By matrix associative property)

$= A(|B| I)\text{adj}A$

$= |B|(AI)\text{adj}A$

$= |B|(A\,\text{adj}A)$

$= |B|(|A| I)$

$= (|B||A| I)$

$= (|A||B|)I$

$= |AB| I$

$= (AB)(\text{adj}(AB))$ (By (1))

$\Rightarrow \boxed{\text{adj}B\,\text{adj}A = \text{adj}(AB)}$

* Orthogonal and Unitary Matrix :-

→ **orthogonal Matrix** :-

A matrix A is said to be orthogonal if $A^T A = I$.

Note: If A is an orthogonal matrix then $A^T A = I$.

$\Rightarrow |A^T A| = |I|$

$\Rightarrow |A^T||A| = 1 \quad (\because |AB| = |A||B|)$

$\Rightarrow |A||A| = 1 \quad (\because |A^T| = |A|)$

$\Rightarrow |A|^2 = 1$

$\Rightarrow |A| = \pm 1$

$\Rightarrow |A| \neq 0$

$\therefore A^{-1}$ exists.

$\therefore$ A is invertible.

and $A^T A = I \Rightarrow \boxed{A^T = A^{-1}}$

$\Rightarrow \boxed{AA^T = I}$

$\therefore$ A is orthogonal iff $A^T A = I = AA^T$

i.e. iff $A^T = A^{-1}$.

→ **Unitary matrix** :-

A matrix A is said to be unitary matrix if $A^\theta A = I$.

Note: If A is unitary then $A^\theta A = I$

$\Rightarrow |A^\theta A| = |I|$

$\Rightarrow |A^\theta||A| = 1$.

$\Rightarrow |\bar{A}||A| = 1 \quad (\because |A^\theta| = |\bar{A}|)$

$\Rightarrow \overline{|A|}\,|A| = 1$

$\Rightarrow |A|^2 = 1 \quad (\because z = x+iy,\ \bar{z} = x-iy \Rightarrow z\bar{z} = |z|^2)$

$\Rightarrow |A| = \pm 1$

$\Rightarrow |A| \neq 0$

$\therefore A^{-1}$ exists.

$\therefore$ A is invertible.

and $A^\theta A = I \Rightarrow \boxed{A^\theta = A^{-1}}$

$\Rightarrow \boxed{AA^\theta = I}$

$\therefore$ A is unitary $\Leftrightarrow A^\theta A = I = AA^\theta$

i.e. $\Leftrightarrow A^\theta = A^{-1}$

* Some Properties of orthogonal and unitary Matrices :-

→ If A, B are n-rowed orthogonal matrices then AB & BA are also orthogonal matrices.

Sol^n :- Since A & B are orthogonal.

$\therefore A^T A = I = AA^T \ \&\ B^T B = I = BB^T$ — (1)

Since A & B are n-rowed square matrices.

$\therefore$ AB is also n-rowed square matrix.

Now $(AB)^T(AB) = B^T A^T (AB)$

$= B^T(A^T A)B$ (By associative prop.)

$= B^T(I_n)B$ (By (1))

$= B^T(I_n B)$

$= B^T B$

$= I_n$ (By (1))

$\therefore$ AB is orthogonal.

Similarly we can prove BA is orthogonal.

→ If A, B be n-rowed unitary matrices then AB and BA are also unitary matrices.

Sol'n :- Since A & B are unitary matrices

$\therefore AA^\theta = I = A^\theta A$ & $BB^\theta = I = B^\theta B$ —— ①

and since A, B are n-rowed square matrices.

$\therefore AB$ is also n-rowed square matrix

Now we have

$(AB)^\theta (AB) = (B^\theta A^\theta) AB$

$= B^\theta (A^\theta A) B$

$= B^\theta (IB)$ (By ①)

$= B^\theta B$

$= I$ (By ①)

$\therefore AB$ is unitary.

Similarly we can prove BA is also unitary.

→ A real matrix is unitary $\Leftrightarrow$ it is orthogonal.

Sol'n :- Let A be a real matrix then

$A^\theta = (\bar{A})' = A'$ —— ①

Since A is unitary.

$\therefore A^\theta A = I$

$\Rightarrow A'A = I$ (by ①)

$\Rightarrow A$ is orthogonal.

Conversely suppose that A is orthogonal.

$\therefore A'A = I$

$\Rightarrow A^\theta A = I$ (by ①)

$\Rightarrow A$ is unitary.

→ If P is orthogonal then P^T and P^{-1} are also orthogonal.

Sol'n :- P is orthogonal

$\therefore P^T P = I \Rightarrow PP^T = I$ —— ①

Now we have

$(P^T)^T P^T = PP^T$

$= I$ (by ①)

$\therefore P^T$ is orthogonal.

Again $(P^{-1})^T (P^{-1}) = (P^T)^{-1} P^{-1}$

$= (PP^T)^{-1}$ $(\because (AB)^{-1} = B^{-1}A^{-1})$

$= I^{-1}$ (By ①)

$= I$

$\therefore P^{-1}$ is unitary.

→ If P is unitary then $\bar{P}, P', P^\theta$ and P^{-1} are also unitary.

Sol'n :- Since P is unitary.

$\therefore P^\theta P = I = PP^\theta$ —— ①

(i) Now we have $(\bar{P})^\theta \bar{P} = [\overline{(\bar{P})}]' \bar{P}$

$= [P'] \bar{P}$

$= [\overline{(\bar{P}')}] \bar{P}$

$= \overline{(P^\theta)}\,\bar{P} = \overline{(P^\theta P)}$

$= \bar{I}$ (by ①)

$= I$.

$\therefore \bar{P}$ is unitary.

(ii) $(P')^{\theta} P' = [(\overline{P'})]' P'$

$= [(\bar{P})']' P' = [P^{\theta}]' P'$

$= [P P^{\theta}]' \quad (\because (AB)^T = B^T A^T)$

$= I'$

$= I$

$\therefore P^T$ is unitary.

(iii) we have $(P^{\theta})^{\theta} P^{\theta} = P P^{\theta} = I$

(iv) $((P^{-1})^{\theta} (P^{-1}) = (P^{\theta})^{-1} P^{-1}$

$= (P P^{\theta})^{-1}$

$= I^{-1} = I$

$\therefore P^{-1}$ is unitary.

→ A real skew-symmetric A satisfies the relation $A^2 + I = 0$ where I is the identity matrix. show that A is orthogonal.

Sol'n :- Given that A is real skew symmetric matrix.

$\therefore A^T = -A$ —— (1)

Also given $A^2 + I = 0$

$\Rightarrow A^2 = -I$ —— (2)

Now we have $AA^T = A(-A)$ (By (1))

$= -A^2$

$= -(-I)$ (By (2))

$= I$.

$\therefore$ A is orthogonal.

→ show that if A is Hermitian and P is unitary then $P^{-1}AP$ is Hermitian.

Sol'n :- Since A is Hermitian

$\therefore A^{\theta} = A$ —— (1)

and P is unitary

$P^{\theta} P = I \Rightarrow P^{\theta} = P^{-1}$ —— (2)

Now we have

$(P^{-1}AP)^{\theta} = P^{\theta} A^{\theta} (P^{-1})^{\theta}$

$= P^{\theta} A^{\theta} (P^{\theta})^{\theta}$ (By (2))

$= P^{\theta} A P$ (By (1))

$= P^{-1} A P$ (By (2))

$\therefore P^{-1}AP$ is Hermitian.

* Submatrix of Matrix :-

Suppose A is any matrix then a matrix obtained by leaving some rows and columns from A is called a submatrix of A.

Note :- The matrix A itself is a submatrix of A because it is obtained from A by leaving no rows or columns.

Minors of a Matrix :-

Let A be an $m \times n$ matrix then the determinant of every square submatrix of A is called a minor of the matrix A.

i.e. If we leave $m-p$ rows and $n-p$ columns from A then the square submatrix of A of order p.

The determinant of this square submatrix is called a p-rowed minor of A.

Ex :- $\begin{bmatrix} 2 & 4 & 1 & 9 & 1 \\ 0 & 5 & 2 & 5 & 2 \\ 3 & -2 & 8 & 1 & 8 \\ 1 & 9 & 7 & 3 & 4 \end{bmatrix}_{4\times 5}$

If we leave two columns and one row from A then we get square submatrix of A of order 3.

$\begin{vmatrix} 2 & 5 & 2 \\ 8 & 1 & 8 \\ 7 & 3 & 4 \end{vmatrix}$, $\begin{vmatrix} 1 & 9 & 1 \\ 2 & 5 & 2 \\ 8 & 1 & 8 \end{vmatrix}$ etc.

are 3-rowed minors of A.

* Rank of Matrix :-

A number r is said to be rank of a matrix A if

(i) there exists at least one minor of order r of the matrix A which is non zero and

(ii) Each minor of order $(r+1)$ of the matrix A is zero (i.e. vanishes)

Note :- (1) The rank of matrix A is an highest non-zero minor of order of the matrix A.

(2) Rank of A is denoted by $\rho(A)$ and is unique.

(3) Every matrix will have a rank.

(4) If A is a matrix of order $m \times n$, then $\rho(A) \leq m$ or n (smaller of the two)

(5) If $\rho(A) = n$ then every minor of order $n+1, n+2$ etc is '0'.

(6) A is a square matrix order $n \times n$
$|A| \neq 0 \Leftrightarrow \rho(A) = n$.

(7) $\rho(I_n) = n$.

(8) A is a matrix of order $m \times n$. If every k^{th} order minor ($k < m, k$ ...) is zero then $\rho(A) < k$.

(9) A is a matrix of order $m \times n$. If there is a minor of order K ($K<m$, $K<n$) which is not zero then $\rho(A) \geq K$.

(10). If A is null matrix then $\rho(A)=0$. Since the rank of every non-zero matrix is ≥ 1, we agree to assign the rank, zero, to every null matrix.

Elementary operations (or) Elementary transformations of a matrix :-

1) Interchange of the ith and jth rows : $R_i \leftrightarrow R_j$ or R_{ij}.

2) Multiplying the ith row by a non-zero scalar K: $R_i \to KR_i$ or $R_i(K)$.

3) Adding to the ith row k times the jth row: $R_i \to R_i + KR_j$. (or) $R_{ij}(K)$.

The corresponding column transformations are respectively.

$C_i \leftrightarrow C_j$, $C_i \to KC_i$, $C_i \to C_i + KC_j$

or C_{ij} or $C_i(K)$ or $C_{ij}(K)$

Echelon Matrix :-

A matrix 'A' is said to be in echelon form iff the number of zeros preceding the non-zero elements of a rows increases row by row. The elements of the last row or rows may be all zeros.

(OR)

A matrix 'A' is said to be in ~~echelon form if~~

(i) The number of zeros before the first non-zero element in a row is less than the numbers of such zeros in the next row.

(ii) The elements of the last row or rows may be all zero.

Note :- (1) The first non-zero elements in the rows of an echelon matrix A are called distinguished elements of A.

Ex :- $\begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 5 & 0 & -7 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are echelon matrices

(2) Triangular matrix is also called echelon form.

Row Reduced Echelon Matrix:-

An echelon ~~matrix~~ is called a row reduced echelon matrix or row canonical form iff the distinguished elements are

each equal to 1 and are the only non-zero elements in their respective columns.

Ex :- $\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a row reduced echelon matrix.

Note :- The rank of a matrix in Echelon form is equal to the number of non-zero rows of the matrix.

Ex :-

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

clearly which is in echelon form

$\therefore$ rank A = the number of non-zero rows of A = 2.

Note :- (1) The rank of the transpose of a ~~matrix~~ is equal to the rank of the original matrix

i.e. $\rho(A) = \rho(A^T)$.

(2). The rank of a matrix every element of which is unity is 1.

→ If A is a non-zero column and B is a non-zero row matrix then show that $\rho(AB) = 1$.

Sol'n :- Let $A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix}_{m\times 1}$ and

$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \end{bmatrix}_{1\times n}$.

then

$$AB = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1n} \\ a_{21}b_{11} & a_{21}b_{12} & \cdots & a_{21}b_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}b_{11} & a_{m1}b_{12} & \cdots & a_{m1}b_{1n} \end{bmatrix}.$$

Since A & B are non-zero matrice

$\therefore$ AB is also non-zero matrix.

The matrix AB will have atleast one non-zero element obtained by multiplying corresponding non-zero elements of A & B.

All the two-rowed minors of AB is obviously zero.

But AB is a non-zero matrix.

$\therefore \rho(AB) = 1$.

Ex :- Let $A = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$; $B = \begin{bmatrix} -4 & 5 & \cdots \end{bmatrix}$

then $AB = \begin{bmatrix} -8 & 10 & 12 \\ -4 & 5 & 6 \\ -12 & 15 & 18 \end{bmatrix}$

Here all two-rowed minors are obviously zero.

But AB is non-zero matrix.

$\therefore \rho(AB) = 1$.

→ If $U = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, find th

values of U and U^2.

Sol'n :- clearly U is in echelon form

The number of non-zero rows in echelon form = 3.

$\therefore \rho(U) = 3$

Now $U^2 = U \cdot U$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ which is in echelon form

the number of non-zero rows in echelon form $= 2$.

$\therefore \rho(U^2) = 2$

* Partitioning of Matrices :-

A matrix may be subdivided into sub-matrices by drawing lines parallel to its rows and columns. So that the elements contained in rectangular blocks are the submatrix elements of the given matrix. This is called partitioning of matrices.

A matrix may be partitioned in many ways and it will be partitioned depending on a situation. One useful partitioning is given below.

Let $A = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 7 & 8 & 10 & 11 & 12 \\ \hline 0 & 8 & 2 & 3 & 1 & 2 \\ \hline 4 & 5 & 1 & 3 & 0 & 0 \\ 2 & 4 & 5 & 1 & 1 & 3 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}$

where $A_{11}, A_{12}, A_{21}, A_{22}, A_{31}, A_{32}$ are the submatrices of orders 2×3, 2×3, 1×3, 1×3, 2×3, 2×3 respectively.

→ One more useful representation of a matrix products is given below:

Let $A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}_{m\times1}$ where each of $R_1, R_2, \dots R_m$ is a matrix of order $1\times n$. (or) Row vectors of A.

and $B = [C_1 \; C_2 \dots C_p]_{1\times p}$ where each of $C_1, C_2, \dots C_p$ is a matrix of order $n\times1$ (or) column vectors of B.

then

$$AB = \begin{bmatrix} R_1C_1 & R_2C_2 & \dots & R_1C_p \\ R_2C_1 & R_2C_2 & \dots & C_2C_p \\ -- & -- & \dots & -- \\ -- & -- & \dots & -- \\ R_mC_1 & R_mC_2 & \dots & C_mC_p \end{bmatrix}_{m\times p}$$

where $R_1C_1, R_1C_2, \dots R_mC_p$ are all matrices each of order 1×1.

* Matrices Partitioned identical for addition :-

Let A & B be two matrices of the same order. If the submatrices A_{ij} of A and B_{ij} of B are of the same order then we say that the matrices are identically partitioned.

Ex: Let

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \end{array}\right] \text{ and}$$

$$B = \left[\begin{array}{ccc|c} 0 & -1 & -2 & -3 \\ -4 & -5 & -6 & -7 \\ \hline -8 & -9 & -10 & -11 \end{array}\right]$$

then A & B are identically partitioned

and $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}_{2\times2}$ $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}_{2\times2}$

$$\therefore A+B = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} \\ A_{21}+B_{21} & A_{22}+B_{22} \end{bmatrix}_{2\times 2}$$

* Matrices Partitioned Conformably for multiplication :—

Let $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$ then AB exists.

Let A be partitioned in any way that the partitioning lines drawn parallel to the rows of B are in the same relative position as the partitioning lines drawn parallel to the columns of A.

The matrices A & B partitioned in the above manner are said to be conformably partitioned for multiplication.

Ex :—

$$A = \left[\begin{array}{ccc|cc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \\ -1 & -2 & -3 & -4 & -5 \end{array}\right]_{4\times 5}$$

→ this line is connection with any rule.

$$B = \left[\begin{array}{c|cc|cc} 0 & -1 & -2 & -3 & -4 \\ -5 & -6 & -7 & -8 & -9 \\ -10 & -11 & -12 & -13 & -14 \\ \hline -15 & -16 & -17 & -18 & -19 \\ -20 & -21 & -22 & -23 & -24 \end{array}\right]_{5\times 5}$$

then

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix}$$

Problems :—

→ If P, Q are non-singular matrices, show that if

$$A = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \text{ then } A^{-1} = \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}$$

Let the inverse of $A = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$ partitioned conformably to pre-multiplication be denoted by $B = \begin{bmatrix} M & R \\ N & S \end{bmatrix}$

Then

$$AB = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}\begin{bmatrix} M & R \\ N & S \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} PM+ON & PR+OS \\ OM+QN & OR+QS \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$\Rightarrow PM+ON = I \Rightarrow PM = I$ ($\because$ O is null)

$PR+OS = 0 \Rightarrow PR = 0$

$OM+QN = 0 \Rightarrow QN = 0$

$OR+QS = I \Rightarrow QS = I$

Now since P is non-singular and $PR=0$.

$$\boxed{\therefore R=0}$$

Also, P is non-singular and $PM=I$

$$\boxed{\therefore M = P^{-1}}$$

Similarly Q is Non-singular & $QN=0$

$$\boxed{\therefore N=0}$$

Also Q is Non-singular & $QS=I$

$$\boxed{\therefore S=Q^{-1}}$$

$$\therefore B = \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}$$

→ Find the inverse of $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ where B, C are non-singular.

Sol'n :- Let the inverse of

$M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ Partitioned

Conformably to pre-multiplication by M, be denoted by $A = \begin{bmatrix} P & R \\ Q & S \end{bmatrix}$

then $MA = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} P & R \\ Q & S \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

$\Rightarrow AP + BQ = I$ —— ①

$AR + BS = 0$ —— ②

$CP = 0$ —— ③

$CR = I$ —— ④

Since C is non-singular & $CP = 0$ & $CR = I$.

$\therefore \boxed{P = 0}$ & $\boxed{R = C^{-1}}$

Now from ①, $A0 + BQ = I$

$\Rightarrow \boxed{BQ = I}$

Since B is non-singular,

$\therefore \boxed{Q = B^{-1}}$

② $\equiv BS = -AR$

$\Rightarrow B^{-1}(BS) = -B^{-1}(AR)$ $(\because |B| \neq 0)$

$\Rightarrow (B^{-1}B)S = -(B^{-1}A)C^{-1}$ $(\because R = C^{-1})$

$\Rightarrow IS = -B^{-1}AC^{-1}$

$\Rightarrow \boxed{S = -B^{-1}AC^{-1}}$

$\therefore M^{-1} = A$

$\Rightarrow \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} = \begin{bmatrix} P & R \\ Q & S \end{bmatrix}$

$= \begin{bmatrix} 0 & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix}$.

→ If A, B, C are non-singular, not necessarily of the same size, show that

$$\begin{bmatrix} A & H & G \\ 0 & B & F \\ 0 & 0 & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}HB^{-1} & A^{-1}HB^{-1}FC^{-1} - A^{-1}GC^{-1} \\ 0 & B^{-1} & -B^{-1}FC^{-1} \\ 0 & 0 & C^{-1} \end{bmatrix}$$

Sol'n :- Let the inverse of

$M = \begin{bmatrix} A & H & G \\ 0 & B & F \\ 0 & 0 & C \end{bmatrix}$ Partitioned

Conformably to pre-multiplication by

A denoted by $\begin{bmatrix} P & S & V \\ Q & T & U \\ R & U & X \end{bmatrix}$

then

$\begin{bmatrix} A & H & G \\ 0 & B & F \\ 0 & 0 & C \end{bmatrix}\begin{bmatrix} P & S & V \\ Q & T & W \\ R & U & X \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$

$\Rightarrow AP + HQ + GR = I$ —— ①

$AS + HT + GU = 0$ —— ②

$AV + HW + GX = 0$ —— ③

$BQ + FR = 0$ —— ④

$BT + FV = I$ —— ⑤

$BW + FX = 0$ —— ⑥

$CR = 0$ —— ⑦

$CU = 0$ —— ⑧

$CX = I$ —— ⑨

Since C is non-singular & $CR=0, CU=0$

$CX = I$

$\therefore \boxed{R=0}, \boxed{U=0}, \boxed{X=C^{-1}}$

(4) $\equiv BQ = 0$ — (10)

$BT = I$ — (11), $BW = -FC^{-1}$ — (12)

Since B is non-singular & $BQ=0$,

$BT = I$.

$\boxed{\therefore Q=0}$, $\boxed{T=B^{-1}}$

(12) $\equiv B^{-1}(BW) = -B^{-1}(FC^{-1})$ $(\because |B| \neq 0)$

$\Rightarrow \boxed{W = -B^{-1}FC^{-1}}$

(1) $\equiv AP + 0 + 0 = I$. $(\because Q=0, R=0)$

$\Rightarrow AP = I$

$\Rightarrow \boxed{P = A^{-1}}$ $(\because A$ is non-singular$)$

(2) $\equiv AS + HB^{-1} + 0 = 0$

$\Rightarrow AS = -HB^{-1}$

$\Rightarrow \boxed{S = -A^{-1}HB^{-1}}$ $(\because |A| \neq 0)$

(3) $\equiv AV + H(-B^{-1}FC^{-1}) + GC^{-1} = 0$

$\Rightarrow AV = HB^{-1}FC^{-1} - GC^{-1}$

$\Rightarrow V = A^{-1}HB^{-1}FC^{-1} - A^{-1}GC^{-1}$

$(\because |A| \neq 0)$

$$\begin{bmatrix} A & H & G \\ 0 & B & F \\ 0 & 0 & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}HB^{-1} & A^{-1}HB^{-1}FC^{-1} - A^{-1}GC^{-1} \\ 0 & B^{-1} & -B^{-1}FC^{-1} \\ 0 & 0 & C^{-1} \end{bmatrix}$$

→ Show that the rank of each of $\begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix}$; $\begin{bmatrix} A_1 & B_1 \\ 0 & 0 \end{bmatrix}$ is at most r:

A_1 being an $r \times r$ order matrix.

Sol'n :— (i). Let $M = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix}$

Since A_1 is an $r \times r$ order matrix.

$\therefore$ The matrix A_2 has r columns.

Now every $(r+1)$ – rowed square submatrix of the matrix M has at least one column of zeros.

$\therefore$ All minors of order $(r+1)$ of the matrix M are zero.

$\therefore \rho(M) \leq r$.

Ex :—
$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 2 & 0 \\ \hline 3 & 4 & 0 \\ 5 & 6 & 0 \\ 5 & 0 & 0 \end{array}\right] \Rightarrow M = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix}$$

Since A_1 is an 2×2 order matrix.

$\therefore$ The matrix A_2 has 2 columns

Now every $(2+1)$-rowed square submatrix of the matrix M has atleast one column of zero.

$\therefore$ All minors of order $(2+1)$ of the matrix M are zero.

$\therefore \rho(M) \leq 2$.

(ii). Let $M = \begin{bmatrix} A_1 & B_1 \\ 0 & 0 \end{bmatrix}$

Since A_1 is an $r \times r$ square matrix,

$\therefore$ The matrix B_1 has also 'r' rows

Now every $(r+1)$ rowed square submatrix of the matrix M has at

least one row of zeros.

$\therefore$ All minors of order $(r+1)$ of the matrix M are zero.

$\therefore \rho(M) \leq r$.

Ex :- Let. $M = \left[\begin{array}{cc|ccc} 1 & 2 & 3 & 5 & 6 \\ 3 & 2 & 4 & 6 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow M = \begin{bmatrix} A_1 & B_1 \\ 0 & 0 \end{bmatrix}$

$\rightarrow$ show that the rank of a matrix doesnot alter on affixing any number of additional rows or columns of zeros.

Soln :- Let A be a matrix of rank 'r'.

Let M be the matrix obtained from the matrix A by affixing some additional rows and columns of zeros.

Let $M = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$

Now every $(r+1)$- rowed minor of the matrix M is either a minor of the matrix A or it will have at least one row (or) one column of zeros.

Since rank of A is r.

$\therefore$ Every $(r+1)$ - rowed minor of the matrix A (if there is any) is equal to zero,

$\therefore$ Every $(r+1)$ -rowed minor of the matrix M is equal to zero.

Since the matrix A has atleast one minor of order 'r' not equal to zero.

$\therefore$ At least one r-rowed minor of the matrix M is not equal to zero

$\therefore \rho(M) = r$.

* Elementary Matrices :-

Definition :- A matrix obtained from a unit matrix by a single elementary transformation is called an elementary matrix or E-matrix.

Ex :- $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are the elementary matrices obtained from I_3 by subjecting it to the elementary operations $C_1 \leftrightarrow C_3$, $R_2 \rightarrow 4R_2$, $R_1 \rightarrow R_1 + 2R_2$ respectively.

we shall use the following symbol to denote the elementary matrices of different types :

1). E_{ij} will denote the E-matrix obtained by interchanging i^{th} and j^{th} rows or i^{th} and j^{th} columns in I.

2) $E_i(K) \Rightarrow$ E-matrix obtained by

multiplying every element of ith row or ith column with k in I.

3). $E_{ij}(k) \Rightarrow$ Elementary matrix obtained by multiplying every element of jth row with k and then adding them to the corresponding elements of ith row in I.

4). $E_{ij}'(k) \Rightarrow$ E-matrix obtained by multiplying every element of jth column with k and then adding them to the corresponding elements of ith column in I.

Properties of Elementary Matrices :-

→ 1. Every elementary matrix is a square matrix.

→ 2. $|E_{ij}^T| = -1$ ($\because |I| = 1$).

→ 3. $|E_i(k)| = k$ where $k \neq 0$

($\because |E_i(k)| = k|I| = k(1) = k$).

Ex:- Let

$$E_i(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ then}$$

$$|E_i(k)| = k|I_3|$$
$$= k(1) = k.$$

→ 4. $|E_{ij}(k)| = 1$

Ex:- Let $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then

$$E_{ij}(2) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore |E_{ij}(2)| = 1$

→ 5. $|E_{ij}'(k)| = 1$

→ 6. Every elementary matrix is non-singular (i.e $|E| \neq 0$).

$\therefore E^{-1}$ exists.

Lemma :-

Every elementary row transformation of a product $C = AB$ be effected by subjecting the pre-factor A to the same row operation.

Proof:- Let A and B be $m \times n$ and $n \times p$ matrices then AB is a matrix of order $m \times p$.

Now let $A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}_{m \times 1}$, $B = [C_1 \; C_2 \cdots C_p]_{1 \times p}$

where $R_1, R_2, \ldots R_m$ denote the row vectors of the matrix A and $C_1, C_2, C_3 \ldots C_p$ denote the column vectors of the matrix B.

$$\therefore AB = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} \begin{bmatrix} C_1 & C_2 & \cdots & C_p \end{bmatrix}$$

$$= \begin{bmatrix} R_1C_1 & R_1C_2 & \cdots & R_1C_p \\ R_2C_1 & R_2C_2 & \cdots & R_2C_p \\ \vdots & \vdots & & \vdots \\ R_mC_1 & R_mC_2 & \cdots & R_mC_p \end{bmatrix}$$

Now if σ denotes any elementary row transformation, then

$(\sigma A)B = \sigma(AB)$.

Ex:– If σ denotes the elementary row-transformation $R_1 \leftrightarrow R_2$ then $(\sigma A)B = \sigma(AB)$.

Theorem :– Every Elementary row transformation of a matrix can be obtained by pre-multiplication with the corresponding elementary matrix.

proof :– Let A be an $m\times n$ matrix and let I_m be a unit matrix.

Now $A = I_m A$

Now let σ be any elementary row transformation to be performed on A.

$$\text{Then } \sigma A = \sigma(I_m A)$$
$$= \sigma(I_m)A$$
$$= EA$$

where E is the elementary matrix corresponding to the row operation σ.

Ex :– Let $A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 7 & 1 \\ 3 & 8 & 4 \end{bmatrix}$

{E-row transformation $A \to B$, $I \to E$, i.e. $EA=B$}

Now the E-transformation $R_1 \to R_1 + 2R_2$ transforms A int B

$$\therefore B = \begin{bmatrix} 7 & 20 & 10 \\ 2 & 7 & 1 \\ 3 & 8 & 4 \end{bmatrix}$$

{E-column transform $A \to B$, $I \to E$, i.e. $AE=B$}

Now Apply the same row-transformation $R_1 \to R_1 + 2R_2$ to the unit matrix I_3. i.e. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore E = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now $EA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 2 & 7 & 1 \\ 3 & 8 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 7 & 20 & 10 \\ 2 & 7 & 1 \\ 3 & 8 & 4 \end{bmatrix} = B.$$

H.W.

Lemma (2) :– Every elementary Column transformation of a product $C = AB$ can be effected by subjecting the post-factor B to the same Column operation.

Sol'n :– If σ denotes any column transformation then $A(\sigma B) = \sigma(AB)$

Theorem :– Every elementary Column transformation of a matrix Can be obtained by post-multiplication

with the corresponding elementary matrix.

Sol'n :- $A = AI_n$

If σ is any column transformation then $\sigma(A) = \sigma(AI_n)$

$= A(\sigma I_n)$

$= AE_1$

* Inverses of Elementary Matrices :-

Theorem $E_{ij}^{-1} = E_{ij}$

Proof :- Let the given matrix E_{ij} be the elementary matrix obtained by interchanging the ith and jth rows of a unit matrix.

If we interchange the ith & jth rows of E_{ij} then we get the unit matrix. $\left\{\begin{array}{l} E_{ij} \longrightarrow I \\ I \longrightarrow E_{ij} \\ \text{i.e. } E_{ij}\cdot E_{ij} = I \end{array}\right\}$

But we know that every elementary row transformation of a matrix can be obtained by pre-multiplying with corresponding elementary matrix.

$\therefore (E_{ij})(E_{ij}) = I$

$\Rightarrow E_{ij}^{-1} = E_{ij}$

$\therefore E_{ij}$ is its own inverse.

Similarly we can show that $E_{ij}^{-1} = E_{ij}$ by the elementary column transformation.

Theorem: $[E_i(K)]^{-1} = E_i\left(\frac{1}{K}\right)$ where $K \neq 0$.

Proof :- The given matrix $E_i(K)$ is the elementary matrix obtained by multiplying ith row by K of a unit matrix.

If we again multiply the ith row of $E_i(K)$ by $\frac{1}{K}$ then we get a unit matrix. $\left\{\begin{array}{l} E_i(K) \longrightarrow I \\ I \longrightarrow E_i(1/K) \end{array}\right\}$.

But we know that every E-row transformation of a matrix can obtained by pre-multiplication with corresponding elementary matrix.

Now pre-multiplying the matrix $E_i(K)$ with the elementary matrix $E_i\left(\frac{1}{K}\right)$

$\therefore E_i\left(\frac{1}{K}\right).E_i(K) = I$

$\therefore [E_i(K)]^{-1} = E_i\left(\frac{1}{K}\right)$; $K \neq 0$

Similarly we can show that $[E_i(K)]^{-1} = E_i\left(\frac{1}{K}\right)$ by the E-column transformation.

$\longrightarrow [E_{ij}(K)]^{-1} = E_{ij}(-K)$ where $K \neq 0$.

Proof :- Let the given matrix $E_{ij}(K)$ is the elementary matrix obtained by multiplying element of jth row by K

and adding to the corresponding element of i^{th} row of a unit matrix.

If we again multiply every element of j^{th} row by $(-K)$ and adding to the corresponding elements of i^{th} row of $E_{ij}(K)$ then we get a unit matrix.

we know that every elementary row-transformation of a matrix can be obtained by pre-multiplying with corresponding elementary matrix.

Now pre-multiplying the matrix $E_{ij}(K)$ with the elementary matrix $E_{ij}(-K)$.

$\therefore E_{ij}(K)\,(E_{ij}(-K)) = I$

$\therefore [E_{ij}(K)]^{-1} = E_{ij}(-K);\ K \neq 0.$

Similarly we can show this by any column transformation.

Note :- (1) The inverse of an elementry matrix is also non-singular.

(2) The inverse any elementary is also an elementary matrix.

Theorem: Elementary transformations don't change the rank of matrix.

(or)

The rank of a matrix is invariant if the matrix is subjected to elementary transformations.

Proof :- Let A be a matrix of rank 'r'

i.e. $\rho(A) = r$

Let B be the matrix obtained from the matrix by the E-transformation.

$\therefore$ All the minors of order $r+1$ will be zero.

Let $|A_0|$ be any $(r+1)$-rowed minor of A.

and $|B_0|$ be any $(r+1)$-rowed minor of B having the same position as $|A_0|$.

Now Case(i) : Interchange the i^{th} & j^{th} rows of a matrix doesnot change the matrix.

Let $R_i \leftrightarrow R_j$ be performed on A.

Then $|A_0|$ will be one of the following three types.

1) $|A_0|$ will remain unchanged.

2) Two of its rows will be interchanged.

3). One of its rows will be interchanged with a row not belong to $|A_0|$.

$\therefore$ Now in (1) $|B_0|=0$, $|A_0|=0$.

in (2) $|B_0|=-|A_0|=-0$

$=0$ and

in (3) $|B_0|$ will be equal in magnitude to some other minor of order $(r+1)$ of A.

$\therefore |B_0|=0$

i.e. all the minors of order $(r+1)$ of B are zero.

$\therefore \rho(B)\leq r \Rightarrow \rho(B)\leq \rho(A)$

Again we can obtain A from B by $R_i \leftrightarrow R_j$ and we can prove that $\rho(A)\leq \rho(B)$

$\therefore \rho(A)=\rho(B)$

Case (ii) Multiplying the ith row by a non-zero scalar K doesnot change the rank of the matrix.

Let $R_i \longrightarrow KR_i$ be performed on A then $|A_0|$ will be one of the following two types:

(1) $|A_0|$ will remain unchanged.

(2) All the elements of one of the rows will be multiplied by K.

$\therefore$ Now in (1), $|B_0|=|A_0|=0$ and

in (2), $|B_0|=K|A_0|=K(0)$

$=0$

$\therefore$ All the minors of order $(r+1)$ of B will be zero.

$\therefore \rho(B)\leq r \Rightarrow \rho(B)\leq \rho(A)$.

Again we can obtain A from B by $R_i \longrightarrow \frac{1}{K}R_i$ and we can prove that

$\rho(A)\leq \rho(B)$

$\therefore \rho(A)=\rho(B)$

Case (iii) Adding to the ith row K times the jth row i.e. $R_i \rightarrow R_i+KR_j$.

Let $R_i \longrightarrow R_i+KR_j$ be performed on A then $|A_0|$ will be one of the following three types.

(1) $|A_0|$ will remain unchanged

(2) The elements of one row of $|A_0|$ will have addition of K times the corresponding elements of another row of $|A_0|$.

(3) The elements of one row of $|A_0|$ will have addition of K times the corresponding elements not belonging to $|A_0|$.

Theorem: The pre-multiplication or post-multiplication by an elementary matrix and as such by any series of elementary matrices, do not change the rank of matrix.

Proof: Let A be a given matrix.

Let E be the elementary matrix which is pre-multiply by A.

If λ be the row operation corresponding to the elementary matrix E then $EA = \lambda A$.

But λA does not change the rank of A.

$$\therefore \rho(EA) = \rho(A).$$

Let $E_1, E_2, E_3 \text{----} E_n$ be 'n' elementary matrices, which are to pre-multiply the matrix A.

Let $\lambda_1, \lambda_2 \text{---} \lambda_n$ be row operations corresponding to the elementary matrices $E_1, E_2 \text{--} E_n$ respectively.

then $E_n . E_{n-1} \text{---} E_2 . E_1 . A = \lambda_n \lambda_{n-1} \text{---} \lambda_2 \lambda_1 A$

But $\lambda_n . \lambda_{n-1} \cdots \lambda_2 . \lambda_1 . A$ donot change the rank of A.

$$\therefore \rho(E_n . E_{n-1} \cdots E_2 . E_1 . A) = \rho(A)$$

$\therefore$ Now in ① $|B_0| = |A_0| = 0$ and

in ② & ③ $|B_0| = |A_0| + k$ ti[mes] another $(r+1)^{th}$ order minor of A.

$$\Rightarrow |B_0| = 0 + k(0)$$

$$= 0$$

$\therefore$ All the $(r+1)$-rowed minors B are zero.

$\therefore \rho(B) \le r \Rightarrow \rho(B) \le \rho(A)$

Again we can obtain A from B by $R_i \to R_i - R_j(k)$ (or) $R_{ij}(-k)$

and we can prove that

$$\rho(A) \le \rho(B)$$

$$\therefore \rho(A) = \rho(B)$$

$\therefore$ By the cases (i), (ii) & (iii) we conclude that elementary row transformations on a matrix donot change the rank of the matrix.

Similarly we prove that elementary column transformations on a matrix donot change the rank.

$\therefore$ Elementary transformations on a matrix donot change the rank of the matrix.

Imp

* Reduction to Normal Form (or) First Canonical form:-

Every non-zero matrix can be reduced to one of the following form I_r,

$\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$, $\left[I_r \mid 0\right]$, $\left[\begin{array}{c} I_r \\ \hline 0 \end{array}\right]$ by a finite number of elementary transformations, where I_r is the unit matrix of order r, called its normal form and r is called the rank of the matrix.

Note:- (1) Not every matrix A can be reduced to normal form by row (column) transformations alone.

Ex:- $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Cannot changed to normal form by row-transformation alone.

and $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 5 & 0 & 0 \end{bmatrix}$ Cannot be changed to normal form by Column transformation alone.

Theorem:- If A be an $m \times n$ matrix of rank 'r', there exist non-singular matrices P & Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

Proof:- Given that $\rho(A) = r$

The matrix A can be transformed to normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by, say 's' number of elementary row-transformations and say 't' number of elementary Column transformations.

we know that every elementary row (column) transformation on A is equivalent to pre (post) - multiplication of A by a suitable elementary matrix.

Now there exist elementary matrices $P_1, P_2, \ldots P_s$ as pre-factors and $Q_1, Q_2, \ldots Q_t$ as post-factors of A such that

$$P_s \cdot P_{s-1} \cdots P_2 \cdot P_1 A \cdot Q_1 \cdot Q_2 \cdots Q_t = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad (1)$$

we know that each elementary matrix is a non-singular matrix and the product of non-singular matrices is also non-singular.

Let $(P_s \cdot P_{s-1} \cdots P_2 \cdot P_1) = P$ and $(Q_1 \cdot Q_2 \cdots Q_t) = Q$

$\therefore$ P & Q are non-singular.

$$\therefore \text{(1)} \equiv PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

1984 Theorem Every Non-singular matrix is a product of elementary matrices.

Proof:- Let A be a non-singular matrix of order $n \times n$.

$$\therefore |A| \neq 0.$$

$$\therefore \rho(A) = n.$$

$\therefore$ It can be reduced to the form I_n by a finite number of elementary row or column transformations.

We know that every elementary row (column) transformation on A is equivalent to pre (post) multiplication of A by a suitable elementary matrix.

Now there exist, say 's' elementary matrices $P_1, P_2 \cdots P_{s-1}, P_s$ as pre-factors and 't' elementary matrices $Q_1, Q_2 \cdots Q_t$ as post factors of A such that

$$(P_s \cdot P_{s-1} \cdots P_2 \cdot P_1) A (Q_1, Q_2 \cdots Q_t) = I_n \quad \text{--- (1)}$$

Since $P_1, P_2, \cdots P_s$ & $Q_1, Q_2, \cdots Q_t$ are non-singular matrices.

$\therefore P_1^{-1}, P_2^{-1}, \cdots P_s^{-1}$ & $Q_1^{-1}, Q_2^{-1} \cdots Q_t^{-1}$ are exist.

Also these inverse matrices are elementary matrices.

Now pre-multiplying successively by $P_s^{-1}, P_{s-1}^{-1} \cdots P_2^{-1}, P_1^{-1}$ and post multiplying successively $Q_t^{-1}, Q_{t-1}^{-1} \cdots Q_2^{-1}, Q_1^{-1}$ for (1) we get.

$$A = P_1^{-1} \cdot P_2^{-1} \cdot P_3^{-1} \cdot P_4^{-1} \cdots P_s^{-1} \cdot I_n \cdot Q_t^{-1} Q_{t-1}^{-1} \cdots Q_2^{-1} \cdot Q_1^{-1}.$$

$$= P_1^{-1} \cdot P_2^{-1} \cdots P_s^{-1} \cdot Q_t^{-1} \cdot Q_{t-1}^{-1} \cdots Q_2^{-1} \cdot Q_1^{-1}$$

$=$ Product of elementary matrices.

Theorem The rank of a matrix does not change by pre-multiplication or post multiplication with a non-singular matrix.

Sol^n:- Let A be a given matrix and P be a non-singular matrix such that PA is possible.

We know that the non-singular matrix P can be expressed as a product of elementary matrices.

Let $P = P_1 \cdot P_2 \cdots P_s$ where $P_1, P_2 \cdots P_s$ are elementary matrices.

$$\therefore PA = P_1 \cdot P_2 \cdots P_{s-1} \cdot P_s A$$

i.e. A is pre-multiplied by s elementary matrices.

i.e. Pre-multiplication of A by s elementary matrices is equivalent

to s elementary row operations on A.

But elementary row operations on A donot change the rank of A.

$$\therefore \rho(PA) = \rho(A)$$

Similarly if Q is a non-singular matrix such that AQ is possible then we can prove that

$$\rho(AQ) = \rho(A)$$

Problems :-

→ Compute the matrix $E_{23} \cdot E_{34}(-1) \cdot E_2(-2) \cdot E_{12}$ for an elementary matrix of order 4.

Sol'n :- Consider $E_{23} E_{34}(-1) E_2(-2) E_{12} I_4 =$

$$E_{23} E_{34}(-1) E_2(-2) E_{12} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= E_{23} \cdot E_{34}(-1) E_2(-2) \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= E_{23} E_{34}(-1) \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= E_{23} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ Compute the following matrix $E_{23}(-1) \cdot E_{31} E'_{24} E_3(2)$ for an elementary matrix of order 4.

Sol'n :- Consider

$E_{23}(-1) \cdot E_{31} \cdot E'_{24} =$ ~~$E_{23}(-1) \cdot E_{31} I_4 E'_{24}$~~

$$= E_{23}(-1) \cdot E_{31} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} E'_{24}$$

$$= E_{23}(-1) E_{31} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= E_{23}(-1) \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$\therefore E_{23}(-1) E_{31} E'_{24} E_3(2)$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

→ Reduce the matrix $A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$ into Echelon form and hence find its rank.

Sol'n :– $A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$

$R_5 \to 5R_5$

$\sim \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 5 & -5 & 10 & 0 \end{bmatrix}$

$R_5 \to R_5 - R_1$

$\sim \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & -8 & -4 & -4 \end{bmatrix}$

$R_3 \to R_3 + 8R_2$

$\sim \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 12 & 4 \end{bmatrix}$

∴ This is in echelon form and the number of non-zero rows is 3.

$\therefore \rho(A) = 3.$

→ Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ to Canonical (normal) form and find its rank.

Sol'n :– $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$

$R_2 \to R_2 + 2R_1, \quad R_3 \to R_3 - R_1$

$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$

$C_2 \to C_2 - 2C_1$

$C_3 \to C_3 - C_1$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$

$R_2 \leftrightarrow R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & -8 \\ 0 & 8 & 5 & 0 \end{bmatrix}$

$C_2 \leftrightarrow C_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -8 \\ 0 & 5 & 8 & 0 \end{bmatrix}$

$R_3 \to R_3 - 5R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -8 \\ 0 & 0 & 18 & 40 \end{bmatrix}$

$C_3 \to C_3 + 2C_2, \quad C_4 \to C_4 + 8C_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 18 & 40 \end{bmatrix}$

$R_3 \to \frac{1}{18} R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{20}{9} \end{bmatrix}$$

$C_4 \to C_4 - \frac{20}{9} C_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\sim [I_3 | 0]$

$\therefore \rho(A) = 3$

→ Find the ranks of $A, B, AB, A+B$ & BA where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$$

→ Reduce the following matrices to normal form and find their ranks.

(i) $A = \begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & -1 & 2 & 3 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

→ Find two non-singular matrices P & Q such that PAQ is in the normal form (i.e. $PAQ = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$)

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

Also find the rank of the matrix A.

Sol'n :- we write $A = I_3 A I_3$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - C_1, \quad C_3 \to C_3 - C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to -\frac{1}{2} R_2, \quad R_3 \to -\frac{1}{2} R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{3}{2} & 0 & -\frac{1}{2} \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore PAQ = \left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right]$$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

and $\rho(A) = 2$.

$\boxed{\text{Note:- } P \,\&\, Q \text{ are not unique.}}$

→ Obtain non-singular matrices P & Q such that PAQ is of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$.

→ obtain non-singular matrices P, Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where $A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$

Imp → Express $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ as a product of elementary matrices.

Sol^n:- Given that

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$R_{21}(-1)$ i.e. $R_2 \to R_2 - R_1$

$R_{31}(-1)$

$$\sim \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_{21}(-3)$, $C_{31}(-3)$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$\therefore \rho(A) = 3$

i.e. $|A| \neq 0$.

$\therefore A$ is non-singular matrix.

Now $R_{21}(-1)$, $R_{31}(-1)$ on A is equivalent to pre-multiplication of A by $E_{21}(-1)$ and $E_{31}(-1)$ elementary matrices.

Also $C_{21}(-3)$ and $C_{31}(-3)$ on A is equivalent to post-multiplicatio of A by $E'_{21}(-3)$ and $E'_{31}(-3)$ elementary matrices.

$\therefore E_{31}(-1) \cdot E_{21}(-1) \cdot A \cdot E'_{21}(-3) \cdot E'_{31}(-3) = \cdot$

$\Rightarrow A = [E_{21}(-1)]^{-1} [E_{31}(-1)]^{-1} \cdot I_3 [E'_{31}(-3)]^{-1} [E'_{21}(-3)]^{-1}$

$= E_{21}(1) \cdot E_{31}(1) \cdot E'_{31}(3) \cdot E'_{21}(3)$

$\left(\because [E_{ij}(k)]^{-1} = E_{ij}(-k) \right)$

$\therefore A =$ a product of Elementary matrices.

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\longrightarrow$ Express the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ as a product of Elementary matrices.

* - Equivalence of Matrices :-

Let A be an $m \times n$ and B be an $m \times n$ matrices. If B is obtained from A by finite number of elementary transformations of A then A is called equivalent to B and is denoted by $A \sim B$.

Note :- The relation $\sim$ defined between two matrices called equivalence matrices.

The following three properties of the relation '$\sim$' in the set M of all $m \times n$ matrices are obvious.

1. Reflexivity : If $A \in M$ then $A \sim A$
2. Symmetry : If $A, B \in M$ such that $A \sim B$ then $B \sim A$.
3. Transitivity : If $A, B, C \in M$ such that $A \sim B$ and $B \sim C$ then $A \sim C$.

$\therefore$ The relation '$\sim$' in the set M of all $m \times n$ matrices is equivalence relation.

Row Equivalence :-

A matrix A is said to be row equivalence to B if B is obtained from A by a finite number of E-row transformations of A and is denoted by $A \overset{R}{\sim} B$.

Column Equivalence :-

A matrix A is said to be column equivalent to B if B is obtained from A by a finite number of E-column transformations of A and is denoted by $A \overset{C}{\sim} B$.

$\longrightarrow$ If $A \sim B$ then $\rho(A) = \rho(B)$

Proof :- Since $A \sim B$ i.e. $A \overset{R}{\sim} B$ or $A \overset{C}{\sim} B$

$\therefore$ B is obtained from A by a finite number of elementary transformations of A.

We know that E-transformations do not change the rank of the matrix

$\therefore$ If $A \sim B$ then

$$\rho(A) = \rho(B)$$

Note: If A & B are equivalent matrices then there exist non-singular matrices P & Q such that $B = PAQ$.

→ If A and B are same order and $\rho(A) = \rho(B)$ then $A \sim B$.

Sol'n – Let A & B be two $m \times n$ matrices of the same rank r then

$$A \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

By the symmetry of the equivalence relation.

$$B \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \sim B$$

Now by the transitivity of the equivalence relation

$$A \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \sim B.$$

$$\Rightarrow A \sim B.$$

→ Is the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$ equivalent to I_3.

Sol'n :– Given that $A_3 = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$

Now $|A| = 1(-2) + 1(-6-1) + 2(4)$
$= -2 - 7 + 8$
$= -1$
$\neq 0$

$\therefore \rho(A) = 3.$

But $\rho(I_3) = 3$

$\therefore A \& I_3$ are same order.

and $\rho(A) = \rho(I_3)$

$\therefore A \sim I_3$.

→ Is the pair of matrices $\begin{bmatrix} 4 & 0 & 2 \\ 3 & 1 & -2 \\ -5 & 0 & 0 \end{bmatrix}$; $\begin{bmatrix} 3 & 9 & 0 & 2 \\ 7 & -2 & 0 & 1 \\ 8 & 7 & 4 & 5 \end{bmatrix}$ equivale

Sol'n :– Given matrices are –

$$\begin{bmatrix} 4 & 0 & 2 \\ 3 & 1 & -2 \\ -5 & 0 & 0 \end{bmatrix}_{3\times 3} \text{ and } \begin{bmatrix} 3 & 9 & 0 & 2 \\ 7 & -2 & 0 & 1 \\ 8 & 7 & 4 & 5 \end{bmatrix}_{3\times 4}$$

These two matrices are of differe orders.

$\therefore$ They are not equivalent.

→ Reduce the matrix $A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ to a matrix B by only row transformations and obtain the rank A by inspection of B

Sol'n :– $A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 2 & 4 & 0 \\ 0 & 2 & 3 \\ 3 & 0 & 1 \end{bmatrix}$$

$R_1 \to \frac{1}{2}R_1$,

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 3 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 0 & -6 & 1 \end{bmatrix}$$

$R_2 \to \frac{1}{2}R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3/2 \\ 0 & -6 & 1 \end{bmatrix}$$

$R_1 \to R_1 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3/2 \\ 0 & -6 & 1 \end{bmatrix}$$

$R_3 \to R_3 + 6R_2$

$$\sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3/2 \\ 0 & 0 & 10 \end{bmatrix}$$

$R_3 \to \frac{1}{10}R_3$ $\quad R_1 \to R_1 + 3R_3$, $R_2 \to R_2 - \frac{3}{2}R_3$

$$\sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B \text{ (say)}$$

$\therefore \rho(A) = \rho(B) = 3.$

→ obtain an equivalent matrix for the given matrix

$$A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$ and hence find its rank.

Sol'n :-

$$A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

$C_1 \leftrightarrow C_2$

$$\sim \begin{bmatrix} 1 & 6 & 3 & 8 \\ 2 & 4 & 6 & -1 \\ 3 & 10 & 9 & 7 \\ 4 & 16 & 12 & 15 \end{bmatrix}$$

$C_2 \to C_2 - 6C_1$, $C_3 \to C_3 - 3C_1$

$C_4 \to C_4 - 8C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -8 & 0 & -17 \\ 3 & -8 & 0 & -17 \\ 4 & -8 & 0 & -17 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 4R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -8 & 0 & -17 \\ 0 & -8 & 0 & -17 \\ 0 & -8 & 0 & -17 \end{bmatrix}$$

$C_2 \to -\frac{1}{8}C_2$

$C_4 \to -\frac{1}{17}C_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$C_4 \to C_4 - C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = B \text{ (say)}$$

$$\rho(A) = 2 = \rho(B)$$

Imp

Theorem :—

If A be an $m \times n$ matrix of rank r then there exists a non-singular matrix P such that $PA = \begin{bmatrix} G \\ 0 \end{bmatrix}$, where G is an $r \times n$ matrix of rank 'r' and 0 is a zero-matrix of order $(m-r) \times n$.

Proof :- Given that A is an $m\times n$ matrix of rank 'r'

$\therefore \exists$ non-singular matrices P & Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ — (1)

We know that every non-singular matrix can be expressed as a product of elementary matrices.

Now let $Q = Q_1 \cdot Q_2 \cdots Q_t$ where $Q_1, Q_2, \cdots Q_t$ are elementary matrices

$\therefore$ (1) $\equiv PA \cdot Q_1 \cdot Q_2 \cdots Q_t = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ — (2)

Now every elementary column transformation of a matrix is equivalent to post-multiplication with the corresponding elementary matrix.

Clearly no column transformation can change the last $(m-r)$ rows of a matrix $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ in (2)

($\because$ each element of $(m-r)$ rows is zero)

Now post-multiplying (2) successively by elementary matrices $Q_t^{-1}, Q_{t-1}^{-1} \cdots Q_1^{-1}$

we have $PA = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \cdot Q_t^{-1} \cdot Q_{t-1}^{-1} \cdots Q_2^{-1} \cdot Q_1^{-1}$

$$\Rightarrow PA = \begin{bmatrix} G \\ 0 \end{bmatrix}$$

where G is of order $r \times n$ and 0 is of order $(m-r) \times n$.

Since elementary operations do not change the rank.

$$\rho(I_r) = r = \rho(G).$$

$\therefore$ There exists a non-singular matrix P such that $PA = \begin{bmatrix} G \\ 0 \end{bmatrix}$ where G is an $r \times n$ and is of rank 'r' and 0 is a zero matrix of order $(m-r) \times n$.

Ex:- Suppose $PAQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ is obtained from $I_{3\times 3} A_{3\times 4} I_{4\times 4} = A_{3\times 4}$

$\Rightarrow PA = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} Q_2^{-1} Q_1^{-1}$ where

$Q_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $Q_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$= \begin{bmatrix} G \\ 0 \end{bmatrix}$ where G is of order 2×4 and 0 is of order $(3-2)\times 4$.

$\rho(I_2) = 2 = \rho(G)$.

Theorem :-

If A be an $m\times n$ matrix of rank 'r' then there exists a non-singular matrix Q such that $AQ = [H \; 0]$ where H is a matrix of order $m\times r$ and of rank 'r' and 0 is a zero matrix of order $m\times(n-r)$.

Proof :- Given that A is an $m\times n$ matrix of rank 'r'.

Now $\exists$ non-singular matrices P & Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ —— (1)

We know that every non-singular matrix can be expressed as a product of elementary matrices.

Hence let $P = P_s \cdot P_{s-1} \cdots P_2 \cdot P_1$ where $P_1, P_2 \cdots P_s$ are elementary matrices.

$\therefore$ (1) $\equiv PAQ = P_s \cdot P_{s-1} \cdot P_{s-2} \cdots P_2 \cdot P_1 AQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ —— (2)

Now every elementary row transformation of a matrix is equivalent to pre-multiplication with corresponding elementary matrix.

Clearly no row-operation can change the last $(n-r)$ columns of a matrix $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ in (2).

Now pre-multiplying (2) successively by elementary matrices $P_s^{-1}, P_{s-1}^{-1} \cdots P_3^{-1}, P_2^{-1}, P_1^{-1}$ we have

$$AQ = P_1^{-1} \cdot P_2^{-1} \cdot P_3^{-1} \cdots P_s^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow AQ = [H \; 0]$$

where H is of order $m\times r$ and 0 is a zero matrix of order $m\times(n-r)$.

Since E - transformations do not change the rank.

$\rho(I_r) = r = \rho(H)$

$\therefore \exists$ a non-singular matrix Q such that $AQ = [H \ \ 0]$.

where H is an $m \times r$ matrix of rank 'r' and O is a zero-matrix of order $m \times (n-r)$.

* Rank of a Product of Matrices :-

Theorem :- The rank of a product of two matrices cannot exceed the rank of either matrix.

(or)

If A, B are matrices conformable for multiplication, then $\rho(AB) \le \rho(A)$ and $\rho(AB) \le \rho(B)$.

Proof :- Let A & B be two matrices of orders $m \times n$ and $n \times p$ respectively.

Let $\rho(A) = r_1$, $\rho(B) = r_2$ and $\rho(AB) = r$.

we know that $\exists$ a non-singular matrix P such that

$PA = \begin{bmatrix} G \\ O \end{bmatrix}$, where G is of order $r_1 \times n$ and O is a zero matrix of order $(m - r_1) \times n$.

Now by post-multiplying both sides by B we have

$PAB = \begin{bmatrix} G \\ O \end{bmatrix} B$

$\therefore \rho(PAB) = \rho(AB) = r$.

$\therefore$ rank of the matrix $\begin{bmatrix} G \\ O \end{bmatrix} B = r$.

Since the matrix G has only r_1 non-zero rows.

$\therefore \begin{bmatrix} G \\ O \end{bmatrix} B$ cannot have more than r_1 non-zero rows.

$\therefore$ Rank of the matrix $\begin{bmatrix} G \\ O \end{bmatrix} B \le r_1$.

$\Rightarrow r \le r_1$

i.e. $\rho(AB) \le \rho(A)$ —— (1). (i.e. A is the pre-factor).

Again $\rho(AB) = [\rho(AB)']$

$= \rho[B'A']$

$\le \rho(B')$ (using (1) $\rho(AB) \le \rho(A)$)

$= \rho(B)$ $(\because \rho(B') = \rho(B))$

$= r_2$

$\therefore r_1 \le r_2$.

i.e. $\rho(AB) \le \rho(B)$ —— (2)

$\therefore$ from (1) & (2) we have

$\rho(AB) \le \rho(A)$ and $\rho(AB) \le \rho(B)$.

Imp working Rule for finding the inverse of a non-singular matrix by E-row transformations

Let $A_{n\times n}$ be a non-singular matrix then $A = I_n A$

Now we go on applying E-row transformations only to the matrix A and prefactor I_n of the product $I_n A$ till we reach the result $I_n = BA$.

then B is the inverse of A.

Problems

2009 → Find the inverse of the matrix given below using E-row operations only: $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Sol'n :- Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ then

$A = I_3 A$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to 3R_1$

$$\sim \begin{bmatrix} 6 & 0 & -3 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - R_2$

$$\sim \begin{bmatrix} 1 & -1 & -3 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 + R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 5R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 15 & -6 & 6 \end{bmatrix} A$$

$R_3 \to \frac{1}{3} R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A$$

$I_3 = BA$

where $B = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

$$\therefore A^{-1} = B = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

(or)

Now $I_3 A^{-1} = (BA) A^{-1}$

$\Rightarrow A^{-1} = B(AA^{-1})$

$= BI_3$

$\boxed{A^{-1} = B}$

$$\therefore A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

→ Reduce the matrix

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 1 \end{bmatrix} \text{ to } I_3 \text{ by}$$

E-row transformations only.

→ Compute the inverses of matrices.

$$\text{(i)} \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

using elementary operations.

→ Find the value of α for which the matrix $\begin{bmatrix} 0 & 1 & \alpha \\ 1 & \alpha & 0 \\ \alpha & 0 & 1 \end{bmatrix}$ is invertible and find its inverse.

Sol^n :— Let $A = \begin{bmatrix} 0 & 1 & \alpha \\ 1 & \alpha & 0 \\ \alpha & 0 & 1 \end{bmatrix}$

then $A = I_3 A$

$$\Rightarrow \begin{bmatrix} 0 & 1 & \alpha \\ 1 & \alpha & 0 \\ \alpha & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \longleftrightarrow R_2$

$$\Rightarrow \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & \alpha \\ \alpha & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \longrightarrow R_1 - \alpha R_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\alpha^2 \\ 0 & 1 & \alpha \\ \alpha & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\alpha & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \longrightarrow R_3 - \alpha R_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\alpha^2 \\ 0 & 1 & \alpha \\ 0 & 0 & \alpha^3+1 \end{bmatrix} = \begin{bmatrix} -\alpha & 1 & 0 \\ 1 & 0 & 0 \\ \alpha^2 & -\alpha & 1 \end{bmatrix} A$$

If $-\alpha^2 = 0$, $\alpha = 0$

$\alpha^3 + 1 = 1$ then

$\alpha = 0$ and A is invertible.

when ... A ... $\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}$

Imp → Reduce the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 3 & -5 \\ 1 & 1 & 5 \end{bmatrix} \text{ to } I_3 \text{ by a finite}$$

sequence of E-row transformati and express A as a product of elementary matrices. Reduce A^{-1} $R_{21}(-4), R_{31}(-$

Sol^n :— $A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 3 & -5 \\ 1 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -5 \\ 0 & 0 & 5 \end{bmatrix}$

$$R_{23}(1) \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$R_{21}(1) \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad R_2(-1),\ R_3(1/5) \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore A \sim I_3$.

But the E-matrices corresponding to $R_{21}(-4)$, $R_{31}(-1)$, $R_{23}(1)$, $R_{12}(1)$, $R_2(-1)$, $R_3(1/5)$ and $E_{21}(-4)$, $E_{31}(-1)$, $E_{23}(1)$, $E_{12}(1)$, $E_2(-1)$, $E_3(1/5)$.

$$\therefore E_3(1/5)\cdot E_2(-1)\cdot E_{12}(1)\cdot E_{23}(1)E_{31}(-1)\cdot E_{21}(-4)A = I_3 \quad \text{——— (1)}$$

$$\Rightarrow A = [E_{21}(-4)]^{-1}[E_{31}(-1)]^{-1}[E_{23}(1)]^{-1}[E_{12}(1)]^{-1}[E_2(-1)]^{-1}\cdot[E_3(1/5)]^{-1} I_3.$$

$$\Rightarrow A = E_{21}(4)\cdot E_{31}(1)\cdot E_{23}(-1)\cdot E_{12}(-1)E_2(1)\cdot E_3(5)\cdot I_3$$

$$\left(\because [E_{ij}(-k)]^{-1} = E_{ij}(k)\right)$$

$$A = E_{21}(4)E_{31}(1)\cdot E_{23}(-1)\cdot E_{12}(-1)\cdot E_2(1)E_3(5)$$

$=$ Product of Elementary matrices.

$$(1) \Rightarrow E_3(1/5)E_2(-1)\cdot E_{12}(1)E_{23}(1)E_{31}(-1)\cdot E_{21}(-4)\, A\cdot A^{-1} = I_3 A^{-1}$$

$$\Rightarrow E_3(1/5)E_2(-1)\cdot E_{12}(1)E_{23}(1)E_{31}(-1)E_{21}(-4)\, I_3 = A^{-1}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & 1 & 1 \\ -5 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}\begin{bmatrix} -4 & 1 & 1 \\ 5 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 1 & 1 \\ 5 & -1 & -1 \\ -1/5 & 0 & 1/5 \end{bmatrix}$$

* Row space of a Matrix :-

Let $A = [a_{ij}]$ be an $m\times n$ matrix then

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

The rows of A are $R_1 = (a_{11}, a_{12}, \cdots a_{1n})$, $R_2 = (a_{21}, a_{22} \cdots a_{2n})$, ---- $R_m = (a_{m1}, a_{m2}, \cdots a_{mn})$. and each of these being an n-tuple over a field F, is a member of the vectorspace F^n (or $V_n(F)$).

The linear span of these vectors i.e. $L(\{R_1, R_2, \cdots R_m\})$ is a subspace of F^n and is called the rowspace of A.

These vectors are called row vectors and is denoted by row sp(A)

i.e. row sp(A) = span$(R_1, R_2, \cdots R_m)$

Similarly the space spanned by the column vectors

i.e. $L(\{C_1, C_2, \cdots C_n\})$ is a subspace of F^n and is called the column space of A.

where $C_1 = (a_{11}, a_{21}, \cdots a_{m1})$
$C_2 = (a_{12}, a_{22}, \cdots a_{m2})$
- - - - - - - -
$C_n = (a_{1n}, a_{2n}, \cdots a_{mn})$ and is denoted by col sp(A).

i.e. col sp(A) = span$(C_1, C_2, \cdots C_n)$

Note :- (1). Column space of A is the same as the row space of A^T.

i.e. col sp(A) = row sp(A^T)

(2) the non-zero rows of an echelon matrix are L.I.

Dimension of row space of A = maximum number of L.I rows of A.

= maximum number of L.I rows of echelon matrix of A.

= number of non-zero rows of an echelon matrix of A.

* Row and Column rank of Matrix

Let $A = [a_{ij}]_{m\times n}$ then the dimension of the row space of A is called the row rank of A and the dimension of the column space of A is called the column rank of A.

→ Find the column of rank of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 4 & 3 & 2 & 3 \\ 3 & 1 & 1 & -4 \\ -1 & -2 & -3 & -7 \\ -4 & -3 & -2 & -8 \end{bmatrix}$$

Sol^n:- we know that Column rank of A is same as row rank of A^T.

Now $A^T = \begin{bmatrix} 1 & 4 & 3 & -1 & -4 \\ 1 & 3 & 1 & -2 & -3 \\ 1 & 2 & -1 & -3 & -2 \\ 2 & 3 & -4 & -7 & -3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 4 & 3 & -1 & -4 \\ 0 & -1 & -2 & -1 & 1 \\ 0 & -2 & -4 & -2 & 2 \\ 0 & -5 & -10 & -5 & 5 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 + 2R_1$

$\sim \begin{bmatrix} 1 & 4 & 3 & -1 & -4 \\ 0 & -1 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$, $R_4 \to R_4 - 5R_2$

which is in echelon form.

~~the~~ number of non-zero rows of this echelon form is 2.

$\therefore \rho(A^T) = 2$

Column rank of $A = 2$.

Note:- 1. The row rank of A is the number of non-zero rows in echelon matrix of A.

2. The row rank and the Column rank of a matrix are equal.

Set — V

* Linear Equations *

→ Now we shall discuss the nature of solutions of a system of non-homogeneous linear Equations.

Now consider the system of m non-homogeneous linear equations in 'n' unknowns $x_1, x_2, x_3, \ldots, x_n$.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad (1)$$

The system (1) can be expressed as the matrix equation $AX = B$ — (2)

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$

→ The matrix A is called the Coefficient matrix.

→ Any set of values of $x_1, x_2, \ldots, x_n$ which simultaneously satisfy all these equations, is called a solution of the system (1). When the system of equations has one or more solutions, the equations are said to be consistent.

otherwise they are said to be inconsistent.

The matrix

$$[A \mid B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{bmatrix}_{m\times(n+1)}$$

is called the augmented matrix of the given system of equations (1).

Imp. *** **Condition for Consistency:**

Theorem :— The equation $AX = B$ is consistent i.e. possesses solution iff the two matrices A & $[A|B]$ are of the same rank.

proof :— we write

$A = [C_1 \; C_2 \; C_3 \cdots C_r \; C_{r+1} \; C_{r+2} \cdots C_n]_{m\times r}$

where $C_1, C_2, \ldots C_n$ are matrices each of order $m \times 1$.

The system $AX = B$ is equivalent to

$$[C_1 \ C_2 \ C_3 \ldots C_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = B$$

$$\Rightarrow x_1 C_1 + x_2 C_2 + x_3 C_3 + \ldots + x_n C_n = B \quad \text{--- (1)}$$

Let $\rho(A) = r$ then A has r Linear Independent columns, without loss of generality.

Let the first r columns of A be LI. i.e. the first r columns $C_1, C_2, \ldots C_r$ form L.I. set.

$\therefore$ Each of the remaining $n-r$ columns is a linear combination of the first r columns $C_1, C_2, \ldots C_r$.

Necessary Condition :-

Let $AX = B$ be Consistent then $\exists$ n Scalars (real numbers) $k_1, k_2, k_3 \ldots k_n$ such that

$$k_1 C_1 + k_2 C_2 + \ldots + k_n C_n = B \quad \text{--- (2)}$$

Let $\rho(A) = r$

Since each of the $(n-r)$ columns $C_{r+1}, C_{r+2}, C_{r+3} \ldots C_n$ is a linear Combination of the first r Columns $C_1, C_2, \ldots C_r$.

$\therefore$ from (2), we have

B is also a linear Combination of $C_1, C_2 \ldots C_r$.

$\therefore$ The maximum number of linearly independent columns of $[A \,\vdots\, B]$ is also r.

$$\therefore \rho(A|B) = r$$

$$\text{i.e. } \rho(A) = \rho(A|B).$$

Sufficient Condition :-

Let $\rho(A) = \rho(A|B) = r$. then the maximum number of linearly independent columns of $[A|B]$ is r and its columns $C_1, C_2 \ldots C_r$.

$\therefore$ B is a linear Combination of $C_1, C_2, \ldots C_r$.

Now $\exists$ scalars $P_1, P_2, \ldots P_r$ such that $P_1 C_1 + P_2 C_2 + \ldots + P_r C_r = B$

i.e. $P_1 C_1 + P_2 C_2 + \ldots + P_r C_r + 0 C_{r+1}$

$$+ 0 C_{r+2} + \ldots + 0 C_n = B \quad \text{--- (3)}$$

Now comparing (1) & (3), we get

$x_1 = P_1, \ x_2 = P_2, \ldots x_r = P_r, x_{r+1} = 0$

$$x_{r+2} = 0 \ldots x_n = 0.$$

$\therefore$ The system $AX = B$ has a solution

$\therefore$ The system is consistent.

Note:— Let $AX = B$ and $CX = D$ be two linear systems; each of m equations in 'n' unknowns.

If the augmented matrices $[A|B]$ & $[C|D]$ of these systems are row equivalent then both linear systems have exactly the same solutions.

If one system has no solution then the other system has no solution.

2000

→ **Theorem:**—

If, A is a non-singular matrix of order 'n' then the linear system $AX = B$ in 'n' unknowns has a unique solution.

Proof:— Since A is a non-singular matrix of order 'n'

$\therefore |A| \neq 0$.

$\Rightarrow \rho(A) = n$ & $\rho(A|B) = n$

$\therefore \rho(A) = \rho(A|B)$

$\Rightarrow AX = B$ is consistent and it has a solution.

Also A^{-1} exists. ($\because |A| \neq 0$)

$\therefore A^{-1}(AX) = A^{-1}B$

$\Rightarrow IX = A^{-1}B$

$\Rightarrow \boxed{X = A^{-1}B}$ is a solution of $AX = B$.

If possible let x_1 & x_2 be two solutions of $AX = B$.

$\therefore Ax_1 = B$ & $Ax_2 = B$

$\Rightarrow Ax_1 = Ax_2$

$\Rightarrow A^{-1}(Ax_1) = A^{-1}(Ax_2)$

$\Rightarrow Ix_1 = Ix_2$

$\Rightarrow \boxed{x_1 = x_2}$

$\therefore$ The solution $x = A^{-1}B$ of $AX = B$ is unique.

* Working rule for finding the solution of equation '$AX = B$':

Suppose the coefficient matrix A is of the type $m \times n$. i.e. we have m equations in 'n' unknowns.

— write the augmented matrix $[A|B]$ and reduce it to an echelon form by applying only Elementary row operations on it.

This echelon form will enable us to know the ranks of the augmented matrix $[A|B]$ and the coefficient matrix A. Then the following cases will arise.

Case(i): If $\rho(A) = \rho(A|B) =$ number of unknowns then the given system of equations is consistent and has unique solution.

Case(ii): If $\rho(A) = \rho(A|B) <$ the number of unknowns then the given system of equations is Consistent and has infinite solutions.

Case(iii): If $\rho(A) \neq \rho(A|B)$ then the given system is not consistent and has no solution.

Problems

→ show that the equations $x+y+z=6$, $x+2y+3z=14$, $x+4y+7z=30$ are Consistent and solve them.

Sol: the matrix form of the given system is

$$AX = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix} = B$$

The augmented matrix.

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - 3R_2$$

∴ which is in echelon form.

∴ $\rho(A|B) = 2$ & $\rho(A) = 2$.

∴ $\rho(A|B) = \rho(A) = 2 <$ the number of three unknown variables x, y, z.

∴ The given system of equations is Consistent and has infinite number of solutions.

Now write the matrix equation with echelon form.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x+y+z \\ y+2z \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

∴ $x+y+z=6$ — (1), $y+2z=8$ — (2)

Now taking $z=t$ in (2) where t is arbitrary constant

$$\boxed{y = 8-2t}$$

∴ (1) $\equiv x = 6-t-8+2t$

$\Rightarrow \boxed{x = t-2}$

∴ $x = t-2$, $y = 8-2t$ and $z = t$

where t is arbitrary constant. constitute the general solution of the given system.

→ Apply the test of rank to examine of the following equations are consistent:

$2x - y + 3z = 8$

$-x + 2y + z = 4$, $3x + y - 4z = 0$ and if consistent, find the complete solution.

Sol'n: Now write the single matrix equation of the given system is

$$AX = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix} = B$$

The augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{array}\right]$$

$R_1 \to R_2$

$$\sim \left[\begin{array}{ccc|c} -1 & 2 & 1 & 4 \\ 2 & -1 & 3 & 8 \\ 3 & 1 & -4 & 0 \end{array}\right]$$

$R_3 \to R_3 + 3R_1$
$R_2 \to R_2 + 2R_1$

$$\sim \left[\begin{array}{ccc|c} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 7 & -1 & 12 \end{array}\right]$$

$R_3 \to 3R_3$

$$\sim \left[\begin{array}{ccc|c} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 21 & -3 & 36 \end{array}\right]$$

$R_3 \to R_3 - 7R_2$

$$\sim \left[\begin{array}{ccc|c} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 0 & -38 & -76 \end{array}\right]$$

$R_3 \to -\frac{1}{38} R_3$

$$\sim \left[\begin{array}{ccc|c} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

∴ which is in echelon form

∴ $\rho(A|B) = 3$ & $\rho(A) = 3$

∴ $\rho(A|B) = \rho(A) = 3$
= the number of unknown variables.

∴ The given equations are consistent and have unique solution.

Again re-write the matrix equation with echelon form is

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \\ 2 \end{bmatrix}$$

$\Rightarrow -x + 2y + z = 4$

$3y + 5z = 16$

$\boxed{z = 2}$

∴ $3y = 16 - 5(2)$
$= 6$

$\Rightarrow \boxed{y = 2}$ and $\boxed{x = 2}$

∴ $x = 2$, $y = 2$, $z = 2$

2004 → Solve the following system of linear equations.

$$x_1 - 2x_2 - 3x_3 + 4x_4 = -1$$
$$-x_1 + 3x_2 + 5x_3 - 5x_4 - 2x_5 = 0$$
$$2x_1 + x_2 - 2x_3 + 3x_4 - 4x_5 = 17.$$

2004 → verify whether the following system of equations are consistent.

$$x + 3z = 5$$
$$-2x + 5y - z = 0$$
$$-x + 4y + z = 4.$$

→ show that the equations $x+y+z=-3$, $3x+y-2z=-2$, $2x+4y+7z=7$ are not consistent.

Solⁿ :- we write the single matrix equation of the system is

$$AX = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix} = B$$

the augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 0 & 20 \end{array}\right]$$

$\therefore \rho(A|B) = 3$ & $\rho(A) = 2$

$\therefore \rho(A|B) \neq \rho(A)$

$\therefore$ the given system of equations is not consistent.

→ show that the equations

$$x + 2y - z = 3$$
$$3x - y + 2z = 1$$
$$2x - 2y + 3z = 2$$
$$x - y + z = -1$$

are consistent and solve them.

→ Investigate for what values of λ and μ the equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ have —
(i) no solution (ii) a unique solution, (iii) infinitely many solutions.

sol'n :- write the matrix equation of the given system.

$$AX = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix} = B$$

The augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

— If $\lambda = 3$ & $\mu \neq 10$ then

$\rho(A|B) = 3$ & $\rho(A) = 2$

$\therefore \rho(A|B) \neq \rho(A)$

$\therefore$ The given equations have no solutions.

— If $\lambda \neq 3$ and $\mu =$ any value then $\rho(A|B) = \rho(A) = 3 =$ the number of unknown variables.

$\therefore$ The equations are consistent and have unique solution.

— If $\lambda = 3$ and $\mu = 10$ then $\rho(A|B) = \rho(A) = 2 <$ the number of unknown variables.

$\therefore$ The given equations are consistent and have infinite solutions.

Prob: For what values of the parameter λ will the following equations fail to have unique solutions. $3x - y + \lambda z = 1$, $2x + y + z = 2$, $x + 2y - \lambda z = -1$.

For the equations have any solution for these values of λ?

write the matrix equation of the given system is

$$AX = \begin{bmatrix} 3 & -1 & \lambda \\ 2 & 1 & 1 \\ 1 & 2 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = B$$

The augmented matrix.

$$[A|B] = \left[\begin{array}{ccc|c} 3 & -1 & \lambda & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & -\lambda & -1 \end{array}\right]$$

$R_1 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -\lambda & -1 \\ 2 & 1 & 1 & 2 \\ 3 & -1 & \lambda & 1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -\lambda & -1 \\ 0 & -3 & 1+2\lambda & 4 \\ 0 & -7 & 4\lambda & 4 \end{array}\right]$$

$R_3 \to -3R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & -\lambda & -1 \\ 0 & -3 & 1+2\lambda & 4 \\ 0 & 21 & -12\lambda & -12 \end{array}\right]$$

$R_3 \to R_3 + 7R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -\lambda & -1 \\ 0 & -3 & 1+2\lambda & 4 \\ 0 & 0 & 2\lambda+7 & 16 \end{array}\right]$$

clearly which is in echelon form

If $2\lambda + 7 \neq 0$ then $\lambda \neq -7/2$

$\therefore \rho(A|B) = \rho(A) = 3$ = the number of unknown variables

$\therefore$ The given system is consistent and has unique solution.

If $\lambda = -7/2$ then

$\rho(A|B) = 3$ & $\rho(A) = 2$

$\therefore \rho(A|B) \neq \rho(A)$

$\therefore$ The given is inconsistent and has no solution.

Imp. For what values of λ the equations
$x+y+z=1$,
$x+2y+4z=\lambda$
$x+4y+10z=\lambda^2$
have a solution ~~and solve them~~ completely in each case.

$\longrightarrow$ Discuss for all values of λ, the system of equations $x+y+4z=6$,
$x+2y-2z=6$,
$\lambda x+y+z=6$, as regards existence and nature of solutions.

Sol^n :- Now, write the matrix ~~equation of the system is~~

$$AX = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & -2 \\ \lambda & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix} = B$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - \lambda R_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 1-\lambda & 1-4\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6-6\lambda \end{bmatrix} \quad \text{---} (1)$$

The given system of equations will have unique solution iff coefficient matrix is non-singular matrix.

i.e. $\begin{vmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 1-\lambda & 1-4\lambda \end{vmatrix} \neq 0$.

$\Rightarrow 1-4\lambda+6-6\lambda \neq 0$

$\Rightarrow \lambda \neq 7/10$

If $\lambda = 7/10$ then the equation (1) becomes.

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & \frac{3}{10} & \frac{-18}{10} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 18/10 \end{bmatrix}$$

$R_3 \to R_3 - 3/10 R_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 18/10 \end{bmatrix}$$

showing that the equations are inconsistent.

* Cramer's Rule of solving a system of 'n' non-homogeneous linear equations in 'n' unknowns:

Let the given system be

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \end{array}\right\}$$

Let
$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0.$$

Let $A_{11}, A_{12}, A_{13}, \cdots$ etc. denote the cofactors of $a_{11}, a_{12}, a_{13} \cdots$ in Δ then multiplying by the given equations respectively by $a_{11}, a_{21}, a_{31}, a_{41} \cdots A_{n_1}$ and adding.

we obtain,

$$x_1(a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} + \cdots + a_{n1}A_{n1}) + x_2(0) + x_3(0) + \cdots + x_n(0) = b_1A_{11} + b_2A_{21} + b_3A_{31} + \cdots + b_nA_{n1}$$

$$\Rightarrow x_1\Delta = \Delta_1$$

where Δ_1 is the determinant obtained by replacing the elements in the first column of Δ by the elements $b_1, b_2, \cdots b_n$.

Similarly
$$x_2\Delta = \Delta_2$$
$$x_3\Delta = \Delta_3$$
$$\cdots$$
$$x_n\Delta = \Delta_n$$

where Δ_i is the determinant obtained by replacing the ith column in Δ by the elements $b_1, b_2 \cdots b_n$.

If $\Delta \neq 0$, then

$$x_1 = \frac{\Delta_1}{\Delta},\ x_2 = \frac{\Delta_2}{\Delta} \cdots x_n = \frac{\Delta_n}{\Delta}.$$

This method of solving n non-homogeneous linear equations in 'n' unknowns is called Cramer's Rule.

Note: If $\Delta = 0$, Cramer's rule of solving is not applicable.

Problems:

→ solve the equations

$$x + y + z = 6;$$
$$x - y + z = 2$$
$$2x - y + 3z = 9$$

Sol'n: (1) By Cramer's Rule: the given system of 3 non-homogeneous linear equations

$x+y+z=6$
$x-y+z=2$
$2x-y+3z=9$

Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 3 \end{vmatrix} = -2 \neq 0$

$\therefore$ By Cramer's Rule

$x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$, $z = \frac{\Delta_3}{\Delta}$

where $\Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ 9 & -1 & 3 \end{vmatrix} = -2,$

$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 2 & 9 & 3 \end{vmatrix} = -4$

$\Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 2 & -1 & 9 \end{vmatrix} = -6$

$\therefore x = \frac{-2}{-2}$, $y = \frac{-4}{-2}$ and $z = \frac{-6}{-2}$

$\Rightarrow x = 1, y = 2$ and $z = 3$.

(2) By Inversion Method :-

The given system can be expressed as $AX = B$.

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 3 \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 2 \\ 9 \end{bmatrix}$

Now $|A| = -2 \neq 0$

$\therefore A^{-1}$ exists.

$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$

$= \frac{1}{-2} \begin{bmatrix} -2 & -4 & 2 \\ -1 & 1 & 0 \\ 1 & 3 & -2 \end{bmatrix}$

Now $AX = B \Rightarrow X = A^{-1}B$

$X = \frac{-1}{2} \begin{bmatrix} -2 & -4 & 2 \\ -1 & 1 & 0 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 9 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\Rightarrow x = 1, y = 2, z = 3.$

(3) The given system can be expressed as

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 9 \end{bmatrix}$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$

$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ -3 \end{bmatrix}$

$R_3 \to R_3 - \frac{3}{2}R_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} x+y+z &= 6 \\ -2y &= -4 \\ z &= 3 \end{aligned}\right\} \text{—— (i)}$$

from (i) $\boxed{z=3}$, $\boxed{y=2}$, $\boxed{x=1}$

Prob. Solve the equations

$\lambda x + 2y - 2z = 1$

$4x + 2\lambda y - z = 2$

$6x + 6y + \lambda z = 3$ Considering specially the case when $\lambda = 2$.

Sol'n: write the matrix equation of the given system.

$$AX = \begin{bmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = B \quad \text{①}$$

The given system of equations will have a unique solution iff the coefficient matrix is non-singular.

i.e. $\begin{vmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{vmatrix} \neq 0$

$\Rightarrow \lambda^3 + 11\lambda - 30 \neq 0$

$\Rightarrow (\lambda-2)(\lambda^2+2\lambda+15) \neq 0$.

Now the only the real root of the equation $(\lambda-2)(\lambda^2+2\lambda+15)=0$ is $\underline{\lambda=2}$.

$\therefore$ If $\lambda \neq 2$ then the given system of equations will have a unique solution given by

$$x = \frac{\begin{vmatrix} 1 & 2 & -2 \\ 2 & 2\lambda & -1 \\ 3 & 6 & \lambda \end{vmatrix}}{\begin{vmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} \lambda & 1 & -2 \\ 4 & 2 & -1 \\ 6 & 3 & \lambda \end{vmatrix}}{\begin{vmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{vmatrix}}$$

$$z = \frac{\begin{vmatrix} \lambda & 2 & 1 \\ 4 & 2\lambda & 2 \\ 6 & 6 & 3 \end{vmatrix}}{\begin{vmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{vmatrix}}$$

In case $\lambda = 2$

$$\text{①} \equiv \begin{bmatrix} 2 & 2 & -2 \\ 4 & 4 & -1 \\ 6 & 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\Rightarrow \begin{bmatrix} 2 & 2 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} 2x+2y-2z &= 1 \\ 3z &= 0 \\ 8z &= 0 \end{aligned}$$

$\Rightarrow z = 0, \quad 2x + 2y = 1$

taking $y = c$ in the above where c is arbitrary.

$\therefore 2x = 1 - 2c$

$$\boxed{x = \frac{1-2c}{2}}$$

$\therefore x = \frac{1-2c}{2}$, $y = c$, $z = 0$ Constitute the general solution of the given system.

Enpl: Investigate for what values of a, b the equations.

$x + 2y + 3z = 4,$
$x + 3y + 4z = 5,$
$x + 3y + az = b$ have

(i) no solution (ii) a unique solution (iii) an infinite number of solutions.

$\longrightarrow$ solve completely the equations

$2x + 3y + z = 9,$
$x + 2y + 3z = 6,$
$3x + y + 2z = 8.$

Homogeneous Linear Equations:—

Let
$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}\right\} \rightarrow (1)$$

be a system of m homogeneous equations in 'n' unknowns $x_1, x_2 \cdots x_n$ then the system (1) can be expressed as the matrix equation $AX = O$. $\rightarrow (2)$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$, $O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1}$

$\Rightarrow$ the matrix A is called the Coefficient matrix.

Now $x_1 = 0, x_2 = 0 \cdots x_n = 0$

i.e. $X = O$ is a solution of (1).

$\therefore$ this solution is called the trivial solution $AX = O$.

— Also the trivial solution is called the zero solution and any other solution is called non-trivial solution (i.e. non-zero solution)

$\therefore Ax=0$ is always consistent.

Imp. If $AX=0$ is a homogeneous system and x_1, x_2 are two solutions of (1) then the solution set of (1) is a subspace of ~~$V_n(F)$, vectorspace~~ of n-tupples over F.

Sol^n :- Given that x_1 & x_2 are two solutions of $AX=0$ —— (1)

$\therefore AX_1=0$ & $AX_2=0$ —— (2)

Now for $K_1, K_2 \in F$

we have $K_1(AX_1)=0$ & $K_2(AX_2)=0$

$\Rightarrow A(K_1x_1)=0$ and $A(K_2X_2)=0$

$\Rightarrow A[K_1X_1+K_2X_2]=0$

$\therefore K_1X_1+K_2X_2$ is also a solution of $AX=0$.

$\therefore$ the solution set of $AX=0$ is a subspace of $V_n(F)$

Imp. The number of L.I solutions of the linear system $AX=0$ is $n-r$; r being the rank of the matrix $A_{m\times n}$.

Proof :- Let

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \text{ and } X=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

Since the rank of the coefficient matrix A is r.

It has r L.I columns.

without loss of generality we can suppose that the first r columns from the left of the matrix A are L.I.

Let $A=[C_1\ C_2\ C_3 \cdots C_r\ C_{r+1} \cdots C_n]_{1\times}$

where $C_1, C_2, \cdots C_n$ are the column vectors of the matrix A each of them being an m-vectors.

$\therefore$ The equation $Ax=0$ becomes

$$[C_1\ C_2\ C_3 \cdots C_r\ C_{r+1} \cdots C_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}=0$$

$$\Rightarrow x_1C_1 + x_2C_2 + \cdots + x_rC_r + \cdots + x_nC_n=0 \quad \text{—— (1)}$$

Since each of the vectors $C_{r+1}, C_{r+2}, \cdots C_n$ is a linear combination of the vectors $C_1, C_2 \cdots C_r$.

we have

$$\left.\begin{aligned} C_{r+1} &= K_{11}C_1 + K_{12}C_2 + \cdots + K_{1r}C_r \\ C_{r+2} &= K_{21}C_1 + K_{22}C_2 + \cdots + K_{2r}C_r \\ &\cdots\cdots \\ C_n &= K_{P1}C_1 + K_{P2}C_2 + \cdots + K_{Pr}C_r \end{aligned}\right\} \text{(2)}$$

where K's $\in F$ and $P=n-r$.

i.e.

$$\left.\begin{aligned} & K_{11}C_1 + K_{12}C_2 + \cdots + K_{1r}C_r + (-1)C_{r+1} + 0C_{r+2} + \cdots + 0\cdot C_n=0 \\ & K_{21}C_1 + K_{22}C_2 + \cdots + K_{2r}C_r + 0C_{r+1} + (-1)C_{r+2} + \cdots + 0C_n=0 \\ & \cdots\cdots \\ & K_{P1}C_1 + K_{P2}C_2 + \cdots + K_{Pr}C_r + 0C_{r+1} + 0C_{r+2} + \cdots + (-1)C_n=0 \end{aligned}\right.$$

Comparing ① & ③, we get

$$x_1 = \begin{bmatrix} k_{11} \\ k_{12} \\ \vdots \\ k_{1r} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} k_{21} \\ k_{22} \\ \vdots \\ k_{2r} \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots x_p = x_{n-r} = \begin{bmatrix} k_{p_1} \\ k_{p_2} \\ \vdots \\ k_{p_r} \\ 0 \\ \vdots \\ -1 \end{bmatrix}$$

as $n-r$ solutions of $Ax=0$.

Now we show that these $(n-r)$ solutions form a L.I system.

Let $k_1x_1 + k_2x_2 + \cdots + k_{n-r}x_{n-r} = 0$ for some $k's \in F$. — ④

then equating $(r+1)^{th}, (r+2)^{th}, (r+3)^{th}, \cdots n^{th}$ entries of columns on the two sides of ④, we get,

$-k_1 = 0, -k_2 = 0, \cdots -k_{n-r} = 0.$

$\Rightarrow k_1 = k_2 = \cdots = k_{n-r} = 0$

It follows that $x_1, x_2 \cdots x_{n-r}$ form a $(n-r)$ linearly independent solutions.

Further we show that every solution of $Ax=0$ is some suitable linear combination of these $(n-r)$ solutions.

Let x, with components $x_1, x_2 \cdots x_n$ be a solution of ①

Consider the vector

$x + x_{r+1}x_1 + x_{r+2}x_2 + \cdots + x_{r+(n-r)}x_{n-r}$ — ⑤

Since ⑤ is a linear combination of solutions,

⑤ is also a solution of $Ax=0$.

It is quite obvious that the last $(n-r)$ components of ⑤ are all zero's. i.e.

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ \vdots \\ x_n \end{bmatrix} + x_{r+1}\begin{bmatrix} k_{11} \\ k_{12} \\ \vdots \\ k_{1r} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_{r+2}\begin{bmatrix} k_{21} \\ k_{22} \\ \vdots \\ k_{2r} \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n\begin{bmatrix} k_{p_1} \\ k_{p_2} \\ \vdots \\ k_{p_r} \\ 0 \\ \vdots \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_{r+1}k_{11} + x_{r+2}k_{21} + \cdots + x_nk_{p_1} \\ x_2 + x_{r+1}k_{12} + x_{r+2}k_{22} + \cdots + x_nk_{p_2} \\ \cdots \\ \cdots \\ x_r + x_{r+1}k_{1r} + \cdots + x_nk_{p_r} \\ x_{r+1} + x_{r+1}(-1) + \cdots + x_n(0) \\ \cdots \\ \cdots \\ x_n + x_{r+1}(0) + x_{r+2}(0) + \cdots x_n(-1) \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_{r+1}k_{11} + x_{r+2}k_{21} + \cdots + x_nk_{p_1} \\ x_2 + x_{r+1}k_{12} + x_{r+2}k_{22} + \cdots + x_nk_{p_2} \\ \cdots \\ \cdots \\ x_r + x_{r+1}k_{1r} + \cdots + x_nk_{p_r} \\ \cdots \\ \cdots \\ 0 + 0 + 0 + \cdots + 0 \\ 0 + 0 + 0 + \cdots + 0 \end{bmatrix}$$

Let the first r components (entries) be $y_1, y_2, y_3 \cdots y_r$. Such a vector with components (entries) $y_1, y_2, y_3 \cdots y_r, 0, 0, 0 \cdots 0$ is also solution of $Ax=0$.

$\therefore$ from (1),

$$y_1C_1 + y_2C_2 + \cdots + y_rC_r + 0C_{r+1} + 0C_{r+2} + \cdots + 0C_n = 0.$$

$$\Rightarrow y_1C_1 + y_2C_2 + \cdots + y_rC_r = 0$$

But $C_1, C_2, \cdots C_r$ are L.I

$\therefore y_1=0, y_2=0, \cdots y_r=0$.

$\therefore$ from (5)

$$x + x_{r+1}X_1 + x_{r+2}X_2 + \cdots + x_nX_{n-r} = 0$$

$$\Rightarrow x = (-x_{r+1})X_1 + (-x_{r+2})X_2 + \cdots + (-x_n)X_{n-r}.$$

$\therefore$ Every solution x is a linear combination of the $(n-r)$ linearly independent solutions $x_1, x_2, \cdots x_{n-r}$, where r is the rank of A.

$\therefore$ The set of solutions $\{x_1, x_2, \cdots x_{n-r}\}$ forms a basis for the vectorspace of all solutions of the system of equations $Ax=0$.

*** Working rule for finding the solution of the equation $Ax=0$:**

Let $Ax=0$ be a given system of m homogeneous equations in 'n' variables then the coefficient matrix A is of the type $m \times n$.

— Reduce the coefficient matrix A to echelon form by applying elementary row transformation only.

This echelon form will help us to know the rank of the matrix A.

— Let x_n be the system $Ax=0$ has solutions.

and $\rho(A) \le \min\{m,n\}$. then the following cases will arise.

Case(i): If $\rho(A) = r = n$ (unknowns) then the system has $n-r = n-n = 0$ L.I solution.

i.e. It has no L.I solution and the only solution of $Ax=0$ is the trivial solution $x_1 = x_2 = \cdots = x_n = 0$ (i.e. the zero solution).

Note: A set containing zero vector is always L.D.

Case(ii): If $\rho(A) = r < n \le m$ (or) $r < m \le n$. then the system $Ax=0$

has $(n-r)$ L.I solutions.

In the process of reducing the matrix A to echelon form, $(m-r)$ equations will be eliminated.

Therefore the given system of 'm' equations will be replaced by an equivalent system of 'r' equations in 'n' unknown variables.

Express the values of some 'r' unknowns interms of remaining $(n-r)$ unknown variables. These $(n-r)$ unknowns can be given any arbitrarily chosen values.

In this case the system has infinitely many solutions which form a vector space of dimension $(n-r)$. [illegible]

Problems

→ Solve the system of equations

$$x+y-3z+2w=0$$
$$2x-y+2z-3w=0$$
$$3x-2y+z-4w=0$$
$$-4x+y-3z+w=0$$

Sol'n :- Now write the matrix equation of given system is

$$AX=\begin{bmatrix}1&1&-3&2\\2&-1&2&-3\\3&-2&1&-4\\-4&1&-3&1\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}=O \quad \text{--- (1)}$$

The coefficient matrix

$$A=\begin{bmatrix}1&1&-3&2\\2&-1&2&-3\\3&-2&1&-4\\-4&1&-3&1\end{bmatrix}$$

$R_2\to R_2-2R_1$, $R_3\to R_3-3R_1$, $R_4\to R_4+4R_1$

$$\sim\begin{bmatrix}1&1&-3&2\\0&-3&8&-7\\0&-5&10&-10\\0&5&-15&9\end{bmatrix}$$

$R_2\longleftrightarrow R_3$

$$\begin{bmatrix}1&1&-3&2\\0&-5&10&-10\\0&-3&8&-7\\0&5&-15&9\end{bmatrix}$$

$R_2\longrightarrow -\frac{1}{5}R_2$

$$\sim\begin{bmatrix}1&1&-3&2\\0&1&-2&2\\0&-3&8&-7\\0&5&-15&9\end{bmatrix}$$

$R_4\longrightarrow R_4+5R_2$, $R_3\longrightarrow R_3+3R_2$

$$\sim\begin{bmatrix}1&1&-3&2\\0&1&-2&2\\0&0&2&-1\\0&0&-5&-1\end{bmatrix}$$

$R_4\longrightarrow -\frac{2}{5}R_4$

$$\sim\begin{bmatrix}1&1&-3&2\\0&1&-2&2\\0&0&2&-1\\0&0&2&2/5\end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 7/5 \end{bmatrix}$$

$\therefore$ clearly which is in echelon form

$\therefore \rho(A) = 4$

= the number of four unknowns.

$\therefore$ The given system of equations has the zero solution.

i.e. $x = y = z = w = 0$.

→ solve completely the system of equations $x + 3y - 2z = 0$, $2x - y + 4z = 0$, $x - 11y + 14z = 0$.

Sol^n:- write the matrix equation $AX = O$ where

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

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$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

clearly which is in echelon form

$\therefore \rho(A) = 2 <$ the number of three unknown variables.

$\therefore$ The given system of equation has non-zero solution.

$\therefore$ The given system of equations will have $n - r = 3 - 2 = 1$ L.I solution

Now we take the arbitrary val to $n - r = 3 - 2 = 1$ variable.

and the remaining 2 variables will be expressed interms of these

Again we write matrix equation in echelon form is

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x + 3y - 2z = 0$ — (1)
$-7y + 8z = 0$ — (2).

Let $\boxed{z = k_1}$ then (2) $\equiv \boxed{y = 8/7 k_1}$

where k_1 is arbitrary constant.

and (1) $\equiv \boxed{x = -10/7 k_1}$

$\therefore$ The general solution of (1) & (2) is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10/7 k_1 \\ 8/7 k_1 \\ k_1 \end{bmatrix}$$

$$= k_1 \begin{bmatrix} -10/7 \\ 8/7 \\ 1 \end{bmatrix}$$

$\therefore$ The general solution of $AX = 0$ is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k_1 \begin{bmatrix} -10/7 \\ 8/7 \\ 1 \end{bmatrix}$$ where k_1 is arbitrary constant.

Therefore the solution $\begin{bmatrix} -10/7 \\ 8/7 \\ 1 \end{bmatrix}$ is L.I.

$\left\{\left(-10/7, 8/7, 1\right)^T\right\}$ is a basis of the solution space.

→ Find a basis and the dimension of the solution space 'S' of the linear equations.

$$x + 2y - 2z + 2s - t = 0$$
$$x + 2y - z + 3s - 2t = 0$$
$$2x + 4y - 7z + s + t = 0$$

Sol^n: write the matrix equation of the given system is $AX = 0$.

where $A = \begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 1 & 2 & -1 & 3 & -1 \\ 2 & 4 & -7 & 1 & 1 \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix}$, $O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Now $A = \begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 1 & 2 & -1 & 3 & -1 \\ 2 & 4 & -7 & 1 & 1 \end{bmatrix}$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 + 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -3 & 3 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$

$$\begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ which is in echelon form.

$\therefore \rho(A) = 2 <$ then number of five unknown variables.

$\therefore$ The given system has non-zero solution.

$\therefore$ The given system will have $n - r = 5 - 2 = 3$ L.I. solutions.

and the dimension of the solution space $S = 3$.

Now we take the arbitrary values to $n - r = 5 - 2 \Rightarrow 3$ variables, and the remaining '2' variables will be expressed interms these.

Now we write the matrix equation in echelon form is

$$\begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x+2y-2z+2w-t=0$ —— ①

$z+w-t=0$ —— ②

Let $t=k_1$, $w=k_2$

where k_1, k_2 are arbitiary Constants

then ② $\equiv \boxed{z=k_1-k_2}$

Let $y=k_3$ then ① $\equiv x+2k_3-2(k_1-k_2)+2k_2-k_1=0$

where k_3 arbitrary Constant.

$\Rightarrow x+2k_3-3k_1+4k_2=0$

$\Rightarrow \boxed{x=3k_1-4k_2-2k_3}$

$\therefore$ The general solution of ① & ② is

$$\begin{bmatrix} x\\ y\\ z\\ s\\ t \end{bmatrix} = \begin{bmatrix} 3k_1-4k_2-2k_3\\ k_3\\ k_1-k_2\\ k_2\\ k_1 \end{bmatrix} = \begin{bmatrix} 3k_1-4k_2-2k_3\\ 0k_1+0k_2+k_3\\ k_1-k_2+0k_3\\ 0k_1+k_2+0k_3\\ k_1+0k_2+0k_3 \end{bmatrix}$$

$$= k_1\begin{bmatrix} 3\\ 0\\ 1\\ 0\\ 1 \end{bmatrix} + k_2\begin{bmatrix} -4\\ 0\\ -1\\ 1\\ 0 \end{bmatrix} + k_3\begin{bmatrix} -2\\ 1\\ 0\\ 0\\ 0 \end{bmatrix}$$

where k_1, k_2, k_3 are arbitrary Constants.

$\therefore$ The solutions

$$\begin{bmatrix} 3\\ 0\\ 1\\ 0\\ 1 \end{bmatrix}, \begin{bmatrix} -4\\ 0\\ -1\\ 1\\ 0 \end{bmatrix}, \begin{bmatrix} -2\\ 1\\ 0\\ 0\\ 0 \end{bmatrix} \text{ are L.I.}$$

$\therefore$ The set of solutions $\{(3,0,1,0,1)^T, (-4,0,-1,1,0)^T, (-2,1,0,0,0)^T\}$ form a basis of the solution space 'S' of the given system of equations.

→ H.W solve completely the system of equations $x+y+z=0$, $2x-y-3z=0$, $3x-5y+4z=0$, $x+17y+4z=0$.

→ Find a basis of the solution space W of the system of equations

$x+2y-2z$ [illegible] $2x+4y-6z+5r$ [illegible]

[illegible] $3r+4s=0$

$3x+6y-8z+7r+s=0$.

→ Prove that a necessary and sufficient condition that values, not all zero may be assigned to 'n' variables $x_1, x_2, x_3, x_4, \ldots x_n$ so that 'n' homogeneous equations $a_{i1}x_1+a_{i2}x_2+a_{i3}x_3+\cdots+a_{in}x_n=0$; $(i=1,2,\ldots n)$ hold simultaneously, is that the determinant of the coefficient matrix vanishes.

Sol'n:- The Coefficient matrix is $A=[a_{ij}]_{n\times n}$ and the given System is $AX=0$.

Now the given system has non-zero solution iff $n-r>0$ where $r=\rho(A)$.

i.e. iff $r<n$

i.e. iff $\rho(A)<n$

i.e. iff A is singular

i.e. iff $|A|=0$.

→ Show that the only real value of λ for which the following system of equations has a non-zero solution is 6.

$x+2y+3z=\lambda x$, $3x+y+2z=\lambda y$, $2x+3y+z=\lambda z$.

Sol:- Write the matrix equation of the system is $AX=0$ —— ①

where

$$A=\begin{bmatrix}1-\lambda & 2 & 3\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{bmatrix},\ X=\begin{bmatrix}x\\y\\z\end{bmatrix},\ O=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Here the coefficient matrix A is a square matrix of order 3×3 and the number of unknowns is also 3.

Now the given system $AX=0$ has non-zero solution

iff $3-r>0$ where $r=\rho(A)$

i.e. iff $3>0$

i.e. iff $r<3$

i.e. iff $|A|=0$

i.e. iff $\begin{vmatrix}1-\lambda & 2 & 3\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{vmatrix}=0$

$C_1\to C_1+C_2+C_3$

i.e. iff $\begin{vmatrix}6-\lambda & 2 & 3\\ 6-\lambda & 1-\lambda & 2\\ 6-\lambda & 3 & 1-\lambda\end{vmatrix}=0$

i.e. iff $(6-\lambda)\begin{vmatrix}1 & 2 & 3\\ 1 & 1-\lambda & 2\\ 1 & 3 & 1-\lambda\end{vmatrix}=0$

$R_2\to R_2-R_1$
$R_3\to R_3-R_1$

i.e. iff $(6-\lambda)\begin{vmatrix}1 & 2 & 3\\ 0 & -1-\lambda & -1\\ 0 & 1 & -2-\lambda\end{vmatrix}=0$.

i.e. iff $(6-\lambda)[(1+\lambda)(2+\lambda)+1]=0$

i.e. iff $(6-\lambda)(\lambda^2+3\lambda+3)=0$.

i.e. iff $\lambda=6$, $\lambda=\frac{-3\pm\sqrt{9-12}}{2}$

i.e. iff $\lambda=6$, $\lambda=\frac{-3\pm\sqrt{-3}}{2}$

∴ The only real value of λ for which the given system has a non-trivial solution is 6.

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Eigen Values and Eigen Vectors

Introduction:-

Let $A = [a_{ij}]_{n\times n}$ be a given n-rowed square matrix

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ be a column vector

Now consider the equation

$$AX = \lambda X \quad \text{--- (1)}$$

where λ is a scalar.

It is obvious that the zero vector $X=0$ is a solution of (1) for any value of λ.

→ If I denotes the unit matrix of order n, then the equation (1) may be written as

$$AX = \lambda I X$$

$$\Rightarrow (A-\lambda I)X = 0 \quad \text{--- (2)}$$

The matrix equation (2) represents the following system of n homogeneous equations in n unknowns:

$$\left.\begin{array}{l} (a_{11}-\lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + a_{2n}x_n = 0 \\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n = 0 \end{array}\right\} \text{--- (3)}$$

The coefficient matrix of the equations (3) is $A-\lambda I$. The necessary and sufficient condition for equations (3) to possess a non-zero solution $(X\neq 0)$ is that the coefficient matrix $A-\lambda I$ should be of rank less than the number of unknowns n.

But this non-zero solution exists iff the matrix $A-\lambda I$ is singular

i.e. $|A-\lambda I| = 0$ *

→ Let $A = [a_{ij}]_{n\times n}$ be any n-rowed square matrix and λ an indeterminate.

— The matrix $A-\lambda I$ is called the characteristic matrix of A, where I is the unit matrix of order n.

— The determinant

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

which is an ordinary polynomial in λ of degree n, is called the characteristic polynomial of A.

— The equation $|A-\lambda I| = 0$ is called the characteristic equation of A, and the roots of this equation are called the characteristic roots (or) characteristic values (or) eigen values

(or) latent roots (or) proper roots of the matrix A.

— The set of the eigen values of A is called the spectrum of A.

→ If λ is a characteristic root of the matrix A, then $|A-\lambda I| = 0$.

and the matrix $A-\lambda I$ is singular.

$\therefore \exists$ a non-zero vector x such that

$$(A-\lambda I)x = 0 \text{ (or) } Ax = \lambda x.$$

* Characteristic Vectors :-

— If λ is a characteristic root of an $n\times n$ matrix A, then the non-zero vector x such that $Ax = \lambda x$ is called a characteristic vector (or) eigen vector of A corresponding to the characteristic root λ.

→ Certain relations b/w characteristic roots and characteristic vectors :-

Theorem: λ is a characteristic root of a matrix A iff $\exists$ a non-zero vector x such that $Ax = \lambda x$.

Proof :- Suppose λ is a characteristic root of the matrix A.

$$\therefore |A-\lambda I| = 0$$

and the matrix $A-\lambda I$ is singular.

$\therefore$ The matrix equation $(A-\lambda I)x = 0$ possess a non-zero solution.

i.e. $\exists$ a non-zero vector x such that $(A-\lambda I)x = 0$

$$\Rightarrow Ax = \lambda x.$$

Conversely suppose that there exists a non-zero vector x such that $Ax = \lambda x$

i.e. $(A-\lambda I)x = 0$

Since the matrix equation $(A-\lambda I)x = 0$ possess a non-zero solution.

$\therefore$ The coefficient matrix $A-\lambda I$ is singular.

i.e. $|A-\lambda I| = 0$

$\therefore \lambda$ is a characteristic root of the matrix A.

Theorem: If x is a characteristic vector of a matrix A corresponding to the characteristic value λ, then kx is also a characteristic vector of A corresponding to the same characteristic value λ. Here k is a non-zero scalar.

Proof :- Suppose x is a charateristic vector of A corresponding to the characteristic value λ.

Then $x \neq 0$ and $Ax = \lambda x$ —— (1)

If K is a non-zero scalar then $Kx \neq 0$.

Now we have

$$A(Kx) = K(Ax)$$
$$= K(\lambda x)$$
$$= \lambda(Kx) \quad (\text{from } (1))$$

$\therefore A(Kx) = \lambda(Kx)$.

$\therefore$ a non-zero vector Kx such that $A(Kx) = \lambda(Kx)$

$\therefore$ Kx is a charateristic vector of A corresponding to the characteristic value λ.

$\therefore$ Corresponding to a characteristic value λ, there corresponds more than one characteristic vectors.

Theorem :-

If x is a characteristic vector of a matrix A, then x cannot correspond to more than one characteristic value of A.

Proof :- Let x be a characteristic vector of a matrix A corresponding to two characteristic values λ_1 and λ_2. then $Ax = \lambda_1 x$, & $Ax = \lambda_2 x$

$$\Rightarrow \lambda_1 x = \lambda_2 x$$
$$\Rightarrow (\lambda_1 - \lambda_2) x = 0$$
$$\Rightarrow \lambda_1 - \lambda_2 = 0 \quad (\because x \neq 0)$$
$$\Rightarrow \lambda_1 = \lambda_2$$

2003, **Linear Independence of characteristic vectors corresponding to distinct characteristic roots:-**

Statement :

The characteristic vectors corresponding to distinct characteristic roots of a matrix are linearly independent.

Proof :- Let $x_1, x_2, x_3, \cdots x_m$ be the characteristic vectors of a matrix A corresponding to distinct characteristic values $\lambda_1, \lambda_2, \cdots \lambda_m$.

Then $Ax_i = \lambda_i x_i$; $i = 1, 2, \cdots m$ —— (1)

To prove that the vectors $x_1, x_2 \cdots x_m$ are linearly independent.

If the vectors $x_1, x_2 \cdots x_m$ are linearly dependent.

then we can choose $r\,(1 \leq r < m)$ such that $x_1, x_2, \cdots x_r$ are L.I and $x_1, x_2 \cdots x_r, x_{r+1}$ are L.D.

$\therefore$ we can choose the scalars $K_1, K_2, \cdots K_{r+1}$ not all zeros such that

$k_1x_1 + k_2x_2 + \cdots + k_rx_r + k_{r+1}x_{r+1} = 0$ — ①

$\Rightarrow A(k_1x_1 + k_2x_2 + \cdots + k_rx_r + k_{r+1}x_{r+1}) = A(0)$

$\Rightarrow k_1(Ax_1) + k_2(Ax_2) + \cdots + k_r(Ax_r) + k_{r+1}(Ax_{r+1}) = 0$

$\Rightarrow k_1(\lambda_1 x_1) + k_2(\lambda_2 x_2) + \cdots + k_r(\lambda_r x_r) + k_{r+1}(\lambda_{r+1}x_{r+1}) = 0 \longrightarrow$ ③ (by using ①)

Now ③ $- \lambda_{r+1}$① $\equiv$

$k_1(\lambda_1 - \lambda_{r+1})x_1 + \cdots k_r(\lambda_r - \lambda_{r+1})x_r = 0$ — ④

Since $x_1, x_2, \ldots x_r$ are L.I and $\lambda_1, \lambda_2, \ldots \lambda_r, \lambda_{r+1}$ are distinct

$\therefore k_1 = 0, k_2 = 0, \ldots k_r = 0.$

Putting $k_1 = 0, k_2, \ldots k_r = 0$ in ② we get $k_{r+1}x_{r+1} = 0$

$\Rightarrow k_{r+1} = 0 \; (\because x_{r+1} \neq 0)$

$\therefore$ from ②, $k_1 = 0, k_2 = 0, \ldots k_r = 0, k_{r+1} = 0$

$\therefore$ which is contradiction to our assumption that the scalars $k_1, k_2, \ldots k_r, k_{r+1}$ are not all zeros.

$\therefore$ our assumption that $x_1, x_2 \ldots x_m$ are L.D is wrong.

$\therefore x_1, x_2, x_3 \ldots x_m$ are L.I.

$\therefore x_1, x_2, \ldots x_m$ which corresponds to distinct characteristic roots of A are L.I.

→ show that the characteristic roots of any diagonal matrix are same as its elements in the diagonal.

Sol^n: Let $A = \text{diag}(d_1, d_2, d_3, \ldots d_n)$ then

$$A = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ - & - & - & \cdots & - \\ - & - & - & \cdots & - \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

Its characteristic equation $|A - \lambda I| = 0$

i.e. $$\begin{vmatrix} d_1-\lambda & 0 & 0 & \cdots & 0 \\ 0 & d_2-\lambda & 0 & \cdots & 0 \\ 0 & 0 & d_3-\lambda & \cdots & 0 \\ - & - & - & \cdots & 0 \\ - & - & - & \cdots & 0 \\ 0 & 0 & 0 & \cdots & d_n-\lambda \end{vmatrix} = 0$$

$\Rightarrow (d_1 - \lambda)(d_2 - \lambda) \cdots (d_n - \lambda) = 0$

$\Rightarrow \lambda = d_1, d_2, \ldots d_n$

$\therefore$ the elements in the diagonal of A are its characteristic roots.

→ Prove that the characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ - & - & \cdots & - \\ - & - & \cdots & - \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ be the triangular matrix.

Its characteristic equation is

$|A-\lambda I| = 0$.

i.e. $\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ - & - & - & \cdots & - \\ - & - & - & \cdots & - \\ 0 & 0 & 0 & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$

$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) = 0$

$\Rightarrow \lambda = a_{11}, a_{22}, \cdots, a_{nn}$

$\therefore$ The diagonal elements of A are the characteristic roots of A.

→ Prove that the square matrices A & A^T have the same characteristic values.

Sol^n:- If λ is any scalar then

$(A-\lambda I)^T = A^T - \lambda I^T$

$= A^T - \lambda I$ ——(1)

Now $|(A-\lambda I)^T| = |A^T - \lambda I|$ (by (1))

$\Rightarrow |A-\lambda I| = |A^T - \lambda I|$ $(\because |A| = |A^T|)$

$\Rightarrow |A-\lambda I| = 0$ iff $|A^T - \lambda I| = 0$

i.e. λ is a characteristic value of A $\Leftrightarrow$ λ is a characteristic value of A^T.

→ Show that '0' is a characteristic root of a matrix iff the matrix is singular.

Sol^n:- 0 is a characteristic root of A

$\Leftrightarrow \lambda = 0$ satisfies the equation $|A-\lambda I| = 0$

$\Leftrightarrow |A - 0I| = 0$

$\Leftrightarrow |A| = 0$

$\Leftrightarrow$ A is singular.

Note:-

(1) λ is a characteristic root of a non-singular matrix

$\Rightarrow \lambda \neq 0$.

(2). At least one characteristic root of every singular matrix is zero.

→ If λ is a characteristic root of the matrix A, show that $k+\lambda$ is a characteristic root of the matrix $A+kI$.

Sol^n:- Let λ be a characteristic root of the matrix A and x be a corresponding characteristic vector.

Then $Ax = \lambda x$ ——(1)

Now $(A+kI)x = Ax + k(Ix)$

$= \lambda x + kx$ (by (1))

$= (\lambda + k)x$

Since $x \neq 0$, $\lambda + k$ is a characteristic root of the matrix $A+kI$ and x is

a corresponding characteristic vector.

→ If $\lambda_1, \lambda_2 \dots \lambda_n$ are the characteristic values of n-rowed square matrix A then show that $K\lambda_1, K\lambda_2, K\lambda_3 \dots K\lambda_n$ are the characteristic values of KA.

Sol'n :- Let $K \neq 0$.

Now $|KA - (\lambda K)I| = |K(A - \lambda I)|$

$= K^n |A - \lambda I|$

$\Rightarrow |KA - (K\lambda)I| = 0$ iff $|A - \lambda I| = 0$

i.e. $K\lambda$ is a characteristic value of KA iff λ is a characteristic value of A.

$\therefore K\lambda_1, K\lambda_2, \dots K\lambda_n$ are the characteristic values of KA iff $\lambda_1, \lambda_2 \dots \lambda_n$ are the characteristic value of A.

→ If $\lambda_1, \lambda_2, \dots \lambda_n$ are the characteristic roots of a n-rowed square matrix A and K is a scalar, show that the characteristic roots of $A - KI$ are $\lambda_1 - K, \lambda_2 - K, \dots \lambda_n - K$.

Sol'n :- Since $\lambda_1, \lambda_2 \dots \lambda_n$ are the characteristic roots of the n-rowed square matrix A.

$\therefore \lambda_1, \lambda_2 \dots \lambda_n$ are the n roots of

$|A - \lambda I| = 0$

which is a n^{th} degree equation in λ.

$\therefore (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda) = 0$ — (1)

The characteristic equation of $A - KI$ is

$|(A - KI) - \lambda I| = 0$

i.e. $|A - (K + \lambda)I| = 0$

$\therefore$ from (1),

$[\lambda_1 - (K+\lambda)][\lambda_2 - (K+\lambda)] \dots [\lambda_n - (K+\lambda)] = 0$

$\Rightarrow [(\lambda_1 - K) - \lambda][(\lambda_2 - K) - \lambda] \dots$

$[(\lambda_n - K) - \lambda] = 0$.

$\Rightarrow \lambda = (\lambda_1 - K), (\lambda_2 - K), \dots (\lambda_n - K)$.

which are the characteristic roots of $(A - KI)$.

→ If the characteristic roots of a n-rowed square matrix A are $\lambda_1, \lambda_2, \dots \lambda_n$ then prove that the characteristic roots of A^2 are $\lambda_1^2, \lambda_2^2, \dots \lambda_n^2$.

Sol'n :- Let λ be the roots of A and X be a corresponding characteristic vector of A.

Then $AX = \lambda X$

$\Rightarrow A(AX) = A(\lambda X)$

$\Rightarrow A^2 X = \lambda (AX)$

$\Rightarrow A^2 X = \lambda (\lambda X)$ ($\because AX = \lambda X$)

$\Rightarrow A^2 X = \lambda^2 X$

$\therefore \lambda^2$ is a characteristic root

of the matrix A^2 corresponding to the characteristic vector X of A^2.

$\therefore$ If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the characteristic roots of A, then $\lambda_1^2, \lambda_2^2, \ldots \lambda_n^2$ are the characteristic roots of A^2.

$\rightarrow$ If the matrix A is non-singular, then show that the eigen values of A^{-1} are the reciprocals of the eigen values of A.

Sol^n :- Since A is non-singular

$\therefore A^{-1}$ exists.

Let λ be an eigen value of A and x be corresponding eigen vector of A.

then $Ax = \lambda x$

$$\Rightarrow A^{-1}(Ax) = A^{-1}(\lambda x)$$

$$\Rightarrow x = \lambda(A^{-1}x)$$

$$\Rightarrow A^{-1}x = \frac{1}{\lambda}x$$

$(\because \lambda \neq 0, A$ is non-singular$)$

$\Rightarrow \frac{1}{\lambda}$ is an eigen value of A^{-1} and x is a corresponding eigen vector.

Converse :-

~~Let k be an eigen~~ value of A^{-1}.

A is non-singular.

$\Rightarrow A^{-1}$ is non-singular

and $(A^{-1})^{-1} = A$.

$\therefore \frac{1}{k}$ is an eigen value of A.

$\therefore$ each eigen value of A^{-1} is the reciprocal of an eigen value of A.

$\therefore$ The eigen values of A^{-1} are nothing but the reciprocals of the eigen values of A.

Imp. ⓔ show that the two matrices A, $C^{-1}AC$ have the same characteristic roots.

Sol^n :- Let $B = C^{-1}AC$ then

$$B - \lambda I = C^{-1}AC - \lambda I$$
$$= C^{-1}AC - C^{-1}\lambda I C$$
$$= C^{-1}(A - \lambda I)C$$

$$\therefore |B - \lambda I| = |C^{-1}(A - \lambda I)C|$$
$$= |C^{-1}|\,|A - \lambda I|\,|C|$$
$$= |A - \lambda I| \quad \left(\because |C^{-1}| = \frac{1}{|C|}\right)$$

$\therefore$ B and A

i.e. $C^{-1}AC$ and A have the

Same characteristic equations.

i.e. A and $C^{-1}AC$ have the same characteristic roots.

$\longrightarrow$ The characteristic roots of a Hermitian matrix are real.

Proof :- Suppose A is a Hermitian matrix.

λ is a characteristic root of A.

and X is a corresponding eigen vector then $AX = \lambda X$ ——— (1)

Now pre-multiplying both sides of (1) by X^θ, we get

$X^\theta AX = \lambda X^\theta X$ ——— (2)

Taking the conjugate transpose of both sides of (2), we get

$(X^\theta AX)^\theta = (\lambda X^\theta X)^\theta$

$\Rightarrow X^\theta A^\theta (X^\theta)^\theta = \bar{\lambda} X^\theta (X^\theta)^\theta$

$\Rightarrow X^\theta A^\theta X = \bar{\lambda} X^\theta X$

$(\because (X^\theta)^\theta = X)$

$\Rightarrow X^\theta AX = \bar{\lambda} X^\theta X$ $(\because A^\theta = A$ because A is Hermitian)

——— (3)

$\therefore$ From (2) & (3), we have

$\lambda X^\theta X = \bar{\lambda} X^\theta X$

$\Rightarrow (\lambda - \bar{\lambda}) X^\theta X = 0$ ——— (4)

But $X \neq 0$

$\therefore X^\theta X \neq 0$

(4) $\equiv \lambda - \bar{\lambda} = 0$

$\Rightarrow \lambda = \bar{\lambda}$

$\therefore \lambda$ is real. $(\because \bar{z} = z$ $\therefore z$ is real)

$\longrightarrow$ Prove that the characteristic roots of a real symmetric matrix are all real.

Sol^n :- Let A be a real symmetric matrix.

$\therefore A^T = A$ ——— (1)

Consider $A^\theta = (\bar{A})^T$

$= A^T$ ($\because$ the elements of A are all real)

$= A$ (by (1))

$\therefore A^\theta = A$

$\Rightarrow$ A is a Hermitian matrix.

We know that the characteristic roots of a Hermitian matrix are real.

$\therefore$ the characteristic roots of A are all real.

i.e. the characteristic roots of a real symmetric matrix are all real.

→ Prove that the eigen values (characteristic roots) of a skew-Hermitian matrix are either purely imaginary (or) zero.

Sol'n :- Let A be a skew-Hermitian matrix.

$\therefore A^{\theta} = -A$ ——(1)

Consider $(iA)^{\theta} = \bar{i}A^{\theta}$

$= -i(-A)$ [$\because \bar{i} = -i$ & by (1)]

$= iA$

$\therefore iA$ is a Hermitian matrix.

Let λ be an characteristic root of A.

$\therefore \exists$ a non-zero column matrix X such that $AX = \lambda X$.

$\Rightarrow i(AX) = i(\lambda X)$

$\Rightarrow (iA)X = (i\lambda)X$

$\Rightarrow i\lambda$ is an eigen value of iA, which is Hermitian matrix.

we know that the roots Hermitian matrix are real.

$\therefore i\lambda$ is a real root of iA.

$\Rightarrow \lambda$ is either purely imaginary (or) zero.

→ Prove that the eigen values of a real skew-symmetric matrix are either purely imaginary (or) zero.

Sol'n :- Let A be a skew-symmetric matrix.

$\therefore A^T = -A$ ——(1)

Consider $A^{\theta} = (\bar{A})^T$

$= A^T$ ($\because$ the elements of A are all real).

$= -A$ (by (1))

$\Rightarrow A$ is a skew-Hermitian matrix.

we know that the characteristic roots of a skew-Hermitian matrix are either purely imaginary (or) zero.

$\therefore$ The characteristic roots of A are either purely imaginary (or) zero.

i.e. the characteristic roots of a real skew-symmetric matrix are either purely imaginary (or) zero.

→ Prove that the eigen values (characteristic roots) of a unitary matrix are of unit modules.

Sol'n :- Let A be an unitary matrix.

$\therefore A^{\theta}A = I$ ——(1)

Let λ be a characteristic root of A.

$\therefore \exists$ a non-zero column matrix.

i.e. characteristic vector X such that

$AX = \lambda X$ ——(2)

Taking conjugate transpose on the sides, we get

$(AX)^{\theta} = (\lambda X)^{\theta}$

$\Rightarrow X^{\theta}A^{\theta} = \bar{\lambda}X^{\theta}$ ——(3)

from (2) & (3), we have

$$(x^\theta A^\theta) Ax = (\bar{\lambda} x^\theta)(\lambda x)$$

$$\Rightarrow x^\theta (A^\theta A) x = (\bar{\lambda}\lambda)(x^\theta x)$$

$$\Rightarrow x^\theta (I) x = |\lambda|^2 (x^\theta x) \quad (\text{by } ①)$$

$$(\because \bar{\lambda}\lambda = |\lambda|^2)$$

$$\Rightarrow x^\theta x = |\lambda|^2 (x^\theta x)$$

$$\Rightarrow (1-|\lambda|^2)(x^\theta x) = 0$$

$$\Rightarrow 1-|\lambda|^2 = 0 \quad (\because x \neq 0 \Rightarrow x^\theta x \neq 0)$$

$$\Rightarrow |\lambda|^2 = 1$$

$$\Rightarrow |\lambda| = 1$$

→ Prove that the eigen values of an orthogonal matrix are of unit modulus.

Sol^n:- We know that if the elements of a unitary matrix A are all real, then A is said to be an orthogonal matrix and the eigen values of a unitary matrix are of unit modulus.

∴ Eigen values of an orthogonal matrix also of unit modulus.

→ A real matrix is unitary ⇔ it is orthogonal.

Sol^n:- A be a real matrix then

$$A^\theta = (\bar{A})^T$$

$$= A^T$$

Since A is unitary.

$$\Leftrightarrow A^\theta A = I \Leftrightarrow A^T A = I$$

$$\Leftrightarrow A \text{ is orthogonal.}$$

→ Determine the characteristic roots of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

Sol^n:- The characteristic matrix of

$$A = A-\lambda I = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0-\lambda & 1 & 2 \\ 1 & 0-\lambda & -1 \\ 2 & -1 & 0-\lambda \end{bmatrix}$$

The characteristic polynomial of

$$A = |A-\lambda I|$$

$$= \begin{vmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix} = -\lambda^3 + 6\lambda - 4$$

∴ The characteristic equation of A is $|A-\lambda I| = 0$

$$\Rightarrow -\lambda^3 + 6\lambda - 4 = 0$$

$$\Rightarrow (\lambda-2)(\lambda^2+2\lambda-2) = 0$$

$$\Rightarrow \lambda = 2 \; ; \; -1 \pm \sqrt{3}$$

∴ The characteristic roots of A are $2, -1 \pm \sqrt{3}$.

→ Prove that ±1 can be the only eigen values of an orthogonal matrix.

Sol^n:- We know that the eigen values of an orthogonal matrix are of unit modulus.

Also ±1 are the only real numbers of unit modulus.

$\therefore \pm 1$ are the only real numbers which can be the characteristic roots (or) eigen values of an orthogonal matrix.

→ If A is both real symmetric and orthogonal, prove that all its eigen values are $+1$ or -1.

Sol'n: If A is a real symmetric matrix, then all its eigen values are real.

If A is orthogonal then all its eigen values must be of unit modulus.

Now ± 1 are the only real numbers of unit modulus.

$\therefore$ If A is both real symmetric and orthogonal then all its eigen values are $+1$ or -1.

H.W → Find the characteristic roots of the 2-rowed orthogonal matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and verify that they are of unit modulus.

→ Show that the roots of the equation

$$\begin{vmatrix} a+x & h & g \\ h & b+x & f \\ g & f & c+x \end{vmatrix} = 0$$ are real where a, b, c, f, g, h are real numbers.

→ If $\lambda \in \mathbb{C}$ is an eigen value of a square matrix A, then prove that $\bar{\lambda}$ is an eigen value of A^θ and conversely. (Or)

Prove that eigen values of A^θ are the conjugates of eigen values of A.

Sol'n: We know that λ is an eigen value of a square matrix A

iff $|A-\lambda I| = 0$.

i.e. iff $\overline{|A-\lambda I|} = 0$.

i.e. iff $|\overline{(A-\lambda I)}| = 0 \quad (\because \overline{|P|} = |\bar{P}|)$

i.e. iff $|(\overline{A-\lambda I})^T| = 0 \quad (\because |P| = |P^T|)$

i.e. iff $|(A-\lambda I)^\theta| = 0$

i.e. iff $|A^\theta - \bar{\lambda} I| = 0$

i.e. iff $\bar{\lambda}$ is an eigen value of A^θ.

* The Construction of orthogonal Matrices:

Imp → Suppose S is an n-rowed real skew-symmetric matrix and I is the unit matrix of order n. Then show that

(i) $I-S$ is non-singular.

(ii) $A = (I+S)(I-S)^{-1}$ is orthogonal.

(iii) $A = (I-S)^{-1}(I+S)$

(iv) If X is a characteristic vector of

S corresponding to the characteristic root λ, then X is also a characteristic vector of A and $\frac{(1+\lambda)}{1-\lambda}$ is the corresponding characteristic root.

Sol'n: (i) Since S is a real skew-symmetric matrix.

$\therefore$ The characteristic roots of S are either pure imaginary (or) zero.

$\therefore$ The roots of the equation $|S-\lambda I| = 0$ are pure imaginary (or) zero.

$\therefore$ 1 is not a root of the equation $|S-\lambda I| = 0$.

$\therefore |S-I| \neq 0$.

$\therefore (S-I)$ is non-singular.

$\Rightarrow I-S$ is non-singular.

(ii) Let $A = (I+S)(I-S)^{-1}$

then $A^T = [(I+S)(I-S)^{-1}]^T$

$= [(I-S)^{-1}]^T (I+S)^T$

$= [(I-S)^T]^{-1} (I+S)^T$ —— ①

Since $(I-S)^T = I^T - S^T$

$= I+S$

$(\because S$ is skew-symmetric $S^T = -S)$

and $(I+S)^T = I-S$

$\therefore$ ① $\equiv$

$A^T = (I+S)^{-1}(I-S)$

Now $A^TA = [(I+S)^{-1}(I-S)][(I+S)(I-S)^{-1}]$

$= (I+S)^{-1}(I+S)(I-S)(I-S)^{-1}$

$(\because (I-S)(I+S) = (I+S)(I-S))$

$= I \cdot I$

$= I$.

$\therefore A$ is orthogonal.

(iii) Since $(I+S)(I-S) = (I-S)(I+S)$

Pre-multiplying throughout by $(I-S)^{-1}$ and post-multiplying throughout by $(I-S)^{-1}$, we get,

$(I-S)^{-1}(I+S)(I-S)(I-S)^{-1} = (I-S)^{-1}(I-S)(I+S)(I-S)^{-1}$

$\Rightarrow (I-S)^{-1}(I+S)I = I(I+S)(I-S)^{-1}$

$\Rightarrow (I-S)^{-1}(I+S) = (I+S)(I-S)^{-1}$

$\Rightarrow (I-S)^{-1}(I+S) = A$ (by (ii))

$\Rightarrow \boxed{A = (I-S)^{-1}(I+S)}$

(iv) Suppose λ is a characteristic root of S and X is the corresponding characteristic vector, then

$SX = \lambda X$

$\Rightarrow X + SX = X + \lambda X$

$\Rightarrow (I+S)X = (1+\lambda)X$ —— ①

Similarly $(I-S)X = (1-\lambda)X$ —— ②

Pre multiplying ② throughout by $(I-S)^{-1}$ we get

$(I-S)^{-1}(I-S)X = (1-\lambda)(I-S)^{-1}X$

$\Rightarrow X = (1-\lambda)(I-S)^{-1}X$

$\Rightarrow (1-\lambda)^{-1}X = (I-S)^{-1}X \quad (\because 1-\lambda \neq 0$ i.e. $\lambda \neq 1)$

$\Rightarrow (I-S)^{-1}X = (1-\lambda)^{-1}X$ — ③

Now Pre-multiplying ① throughout by $(I-S)^{-1}$, we get

$(I-S)^{-1}(I+S)X = (1+\lambda)(I-S)^{-1}X$

$\Rightarrow [(I-S)^{-1}(I+S)]X = [(1+\lambda)(1-\lambda)^{-1}]X$ (from ③)

$\therefore$ X is a characteristic vector of $A = (I-S)^{-1}(I+S)$ and $(1+\lambda)(1-\lambda)^{-1}$ is the corresponding characteristic root.

→ If S is a real skew-symmetric matrix then show that $(I+S)$ is non-singular and $(I-S)(I+S)^{-1}$ is orthogonal.

→ If A is an orthogonal matrix with the property that -1 is not a characteristic root, then A is expressible as $(I+S)(I-S)^{-1}$ for some suitable real skew-symmetric matrix S.

Sol'n :- Given that $A = (I+S)(I-S)^{-1}$ — ①

Post-multiplying both sides of ① by $(I-S)$, we get

$A(I-S) = (I+S)$

$\Rightarrow A - AS = I + S$

$\Rightarrow A - I = AS + S$

$\Rightarrow A - I = (A+I)S$ — ②

Since -1 is not a characteristic root of A.

$\therefore |A-(-1)I| \neq 0$

i.e. $|A+I| \neq 0$

i.e. $A+I$ is non-singular.

$\therefore$ Pre-multiplying bothsides of ② by $(A+I)^{-1}$, we get

$(A+I)^{-1}(A-I) = S$ — ③

Since A is orthogonal matrix

$\therefore$ S is skew-symmetric matrix.

$\therefore$ we can easily show that S is a skew-symmetric matrix.

Now we have

$S^T = [(A+I)^{-1}(A-I)]^T$

$= (A-I)^T[(A+I)^{-1}]^T$

$= (A^T - I^T)[A^T + I^T]^{-1}$

$= (A^T - I)[A^T + I]^{-1}$ — ④

Since $(A^T+I)(A^T-I) = (A^T-I)(A^T+I)$

Pre-multiplying throughout by $(A^T+I)^{-1}$ and Post multiplying throughout ....

by $(A^T+I)^{-1}$, we get

$$(A^T+I)^{-1}(A^T+I)(A^T-I)(A^T+I)^{-1} = (A^T+I)^{-1}(A^T-I)(A^T+I)(A^T+I)^{-1}$$

$$\Rightarrow (A^T-I)(A^T+I)^{-1} = (A^T+I)^{-1}(A^T-I)$$

from ④, we get

$$S^T = (A^T-I)(A^T+I)^{-1}$$

$$= (A^T+A^TA)^{-1}(A^T-A^TA)$$

($\because$ A is orthogonal $\Rightarrow A^TA = I$).

$$\Rightarrow S^T = [A^T(I+A)]^{-1}[A^T(I-A)]$$

$$= (I+A)^{-1}(A^T)^{-1}A^T(I-A)$$

$$= (I+A)^{-1} I (I-A) \quad (\because (A^T)^{-1}A^T = I)$$

$$= (I+A)^{-1}(I-A)$$

$$= -(A+I)^{-1}(A-I)$$

$$= -S.$$

$\therefore$ S is a skew-symmetric matrix.

H.W $\rightarrow$ If A is an orthogonal matrix with the property that -1 is not a characteristic root, then A is expressible as $(I-S)(I+S)^{-1}$ for some suitable skew-symmetric matrix.

2005 $\rightarrow$ If S is a skew-Hermitian matrix, show that the matrices $(I-S)$ and $(I+S)$ are both non-singular.

Also show that $A = (I+S)(I-S)^{-1}$ is a unitary matrix.

Sol'n :- Given that 'S' is a skew-Hermitian matrix.

$$\therefore S^\theta = -S \quad \text{——— ①}$$

We know that the eigen values of a skew-Hermitian matrix 'S' are either purely imaginary (or) zero.

$\therefore$ Neither 1 nor -1 is a root of the equation $|S-\lambda I| = 0$.

$$\Rightarrow |S-I| \neq 0 \text{ and } |S+I| \neq 0.$$

$$\Rightarrow |I-S| \neq 0 \text{ and } |I+S| \neq 0$$

$$(\because |-A| \neq 0 \Rightarrow |A| \neq 0).$$

$\therefore$ $I-S$ and $I+S$ are both non-singular matrices.

Now given that $A = (I+S)(I-S)^{-1}$

Consider $A^\theta = [(I+S)(I-S)^{-1}]^\theta$

$$= [(I-S)^{-1}]^\theta (I+S)^\theta$$

$$= [(I-S)^\theta]^{-1}(I^\theta + S^\theta)$$

$$= (I^\theta - S^\theta)^{-1}(I-S) \quad (\text{by ①})$$

$$= (I+S)^{-1}(I-S) \quad (\text{by ①})$$

$\therefore A^\theta A = (I+S)^{-1}(I-S)(I+S)(I-S)^{-1}$

$= (I+S)^{-1}(I+S)(I-S)(I-S)^{-1}$

$[\because (I-S)(I+S) = (I+S)(I-S)]$

$= I.I$

$= I$

$\therefore A^\theta A = I$

$\therefore$ A is a unitary matrix.

→ Show that every unitary matrix A can, by a suitable choice of skew-Hermitian matrix S be expressed as $A = (I+S)(I-S)^{-1}$ provided that, -1 is not a characteristic root of A.

→ If H is a Hermitian matrix show that $A = (I+iH)^{-1}(I-iH)$ is a unitary matrix. Also show that $A = (I-iH)(I+iH)^{-1}$.

Further show that if λ is an eigen value of H, then $\frac{(1-i\lambda)}{(1+i\lambda)}$ is an eigen value of A.

Sol'n :- Since H is a Hermitian matrix.

$\therefore H^\theta = H$ —— ①

We know that the characteristic roots of H are real.

$\therefore$ The roots of the equation $|H-\lambda I| = 0$ are real.

$\therefore$ Neither i nor $-i$ is a root of the equation $|H-\lambda I| \neq 0$.

$\Rightarrow |H-iI| \neq 0$ and $|H+iI| \neq 0$

$\Rightarrow (H-iI)$ is non-singular and $(H+iI)$ is non-singular.

$\Rightarrow (iH+I)$ & $(iH-I)$ are also non-singular.

$\Rightarrow (I+iH)$ & $(I-iH)$ also non-singular $(\because |A| \neq 0 \Rightarrow |-A| \neq 0)$

(i) Given that $A = (I+iH)^{-1}(I-iH)$

Consider $A^\theta = [(I+iH)^{-1}(I-iH)]^\theta$

$= (I-iH)^\theta((I+iH)^{-1})^\theta$

$= [I^\theta - (iH)^\theta][(I+iH)^\theta]^{-1}$

$= (I^\theta - \bar{i}H^\theta)(I^\theta + \bar{i}H^\theta)^{-1}$

$= (I-(-i)H)(I+(-i)H)^{-1}$

$[\because \bar{i} = -i$ & by ①$]$

$= (I+iH)(I-iH)^{-1}$

$A^\theta A = (I+iH)(I-iH)^{-1}(I+iH)^{-1}(I-iH)$

$= (I+iH)(I+iH)^{-1}(I-iH)^{-1}(I-iH)$

$(\because (I+iH)^{-1}(I-iH)^{-1} = (I-iH)^{-1}(I+iH)^{-1})$

$= I.I$

$= I$

$\therefore A^\theta A = I$

$\therefore A$ is a unitary matrix.

(ii) To show $A = (I-iH)(I+iH)^{-1}$

Since $(I-iH)(I+iH) = (I+iH)(I-iH)$

Pre-multiplying throughout by $(I+iH)^{-1}$ and post-multiplying throughout by $(I+iH)^{-1}$, we get

$$(I+iH)^{-1}(I-iH)(I+iH)(I+iH)^{-1} = (I+iH)^{-1}(I+iH)\cdot(I-iH)(I+iH)^{-1}$$

$$\Rightarrow (I+iH)^{-1}(I-iH) = (I-iH)(I+iH)^{-1}$$

$$\Rightarrow A = (I-iH)(I+iH)^{-1}$$

(iii) Suppose λ is an eigen value of H and x is the corresponding eigen vector of H. then

$$Hx = \lambda x$$

$$\Rightarrow iHx = i\lambda x$$

$$\Rightarrow x + iHx = x + i\lambda x$$

$$\Rightarrow (I+iH)x = (1+i\lambda)x \quad \text{—ⓐ}$$

similarly $(I-iH)x = (1-i\lambda)x$ —ⓑ

Pre-multiplying ⓐ throughout by $(I+iH)^{-1}$ we get

$$(I+iH)^{-1}(I+iH)x = (1+i\lambda)(I+iH)^{-1}x$$

$$\Rightarrow x = (1+i\lambda)(I+iH)^{-1}x$$

$$\Rightarrow (1+i\lambda)^{-1}x = (I+iH)^{-1}x$$

$$\Rightarrow (I+iH)^{-1}x = (1+i\lambda)^{-1}x \quad \text{—ⓒ}$$

Now pre-multiplying ⓑ throughout by $(I+iH)^{-1}$ we get

$$(I+iH)^{-1}(I-iH)x = (1-i\lambda)(I+iH)^{-1}x$$

$$\Rightarrow [(I+iH)^{-1}(I-iH)]x = (1-i\lambda)(1+i\lambda)^{-1}x \quad (\text{by ⓒ})$$

$$\Rightarrow [(I+iH)^{-1}(I-iH)]x = [(1-i\lambda)(1+i\lambda)^{-1}]x$$

$\therefore x$ is a characteristic vector of $A = (I+iH)^{-1}(I-iH)$ and $(1-i\lambda)(1+i\lambda)^{-1}$ is the corresponding characteristic root of A.

2003 → If H is any Hermitian matrix, then $A = (H+iI)^{-1}(H-iI) = (H-iI)(H+iI)^{-1}$ is unitary and every unitary matrix can be thus expressed provided, -1, is not a characteristic root of A.

→ Find the eigen roots and corresponding eigen vectors of the matrix $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ —

Sol^n:– The characteristic matrix of $A = A - \lambda I$

$$= \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix}$$

The characteristic polynomial of $A = |A - \lambda I|$

$$= \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix}$$

$= (1-\lambda)(2-\lambda) - 12$

$= 2 - 3\lambda + \lambda^2 - 12$

$= \lambda^2 - 3\lambda - 10$

$= (\lambda+2)(\lambda-5)$

The characteristic equation of A is $|A - \lambda I| = 0$

$\Rightarrow (\lambda+2)(\lambda-5) = 0$

$\Rightarrow \lambda = 5, -2$

$\therefore$ The required eigen roots of A are $-2, 5$.

To find the eigen vectors associated with -2:

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector of A corresponding to the eigen value -2 then

$$(A - (-2)I)X = 0$$

$$\Rightarrow \begin{bmatrix} 1+2 & 4 \\ 3 & 2+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\Rightarrow \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

$\therefore$ clearly the coefficient is in echelon form.

$\therefore$ The rank of the coefficient matrix $= 1$

$\therefore$ there is $2 - 1 = 1$ L.I eigen vector of A corresponding to eigen root -2.

Now from (1),

$$3x_1 + 4x_2 = 0$$

Let $x_2 = k$ where k is a non-zero parameter.

$\therefore x_1 = -4/3\, k$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4/3\,k \\ k \end{bmatrix}$$

$$= k \begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$$

$$= kX_1$$

Here $X_1 = \begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$ is L.I eigen vector of A corresponding to eigen value -2.

and the set of all eigen vectors of A corresponding to the eigen value -2 is given by kX_1.

where k is non-zero parameter.

To find the eigen vectors associated with 5:

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be an eigen vector of A corresponding to the eigen value 5 then

$$(A - 5I)X = 0.$$

$$\Rightarrow \begin{bmatrix} 1-5 & 4 \\ 3 & 2-5 \end{bmatrix} X = 0$$

$$\Rightarrow \begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to \frac{1}{3}R_2$

$$\Rightarrow \begin{bmatrix} -4 & 4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to 4R_2$

$$\Rightarrow \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\Rightarrow \begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{--- (2)}$$

Clearly the coefficient matrix is in echelon form.

$\therefore$ The rank of the coefficient matrix $= 1$.

$\therefore$ There is $2-1=1$ L.I eigen vector of A corresponding to eigen value 5.

Now from (2),

$$-4x_1 + 4x_2 = 0$$

$$\Rightarrow -x_1 + x_2 = 0$$

Let $x_2 = K$

where K is non-zero parameter.

then $x_1 = K$

$$\therefore X = \begin{bmatrix} K \\ K \end{bmatrix} = K\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Here $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is L.I eigen vector of A corresponding to eigen value 5.

and the set of all eigen vectors of A corresponding to the eigen value 5 is given by KX_2 where K is non-zero parameter.

$\rightarrow$ Find all the eigen values and the eigen vectors of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}.$$

Sol'n :- The characteristic matrix of $A = A - \lambda I$

$$= \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{bmatrix}$$

The characteristic polynomial $= |A - \lambda I|$

$$= \begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)[(4-\lambda)(3-\lambda) - 2] - 1[6 - 2\lambda - 2] + 1[2 - 4 + \lambda]$$

$$= (3-\lambda)[12 - 7\lambda + \lambda^2 - 2] - 4 + 2\lambda + \lambda - 2$$

$$= (3-\lambda)[\lambda^2 - 7\lambda + 10] + 3\lambda - 6.$$

$= 3\lambda^2 - 21\lambda + 30 - \lambda^3 + 7\lambda^2 - 10\lambda + 3\lambda - 6$

$= -\lambda^3 + 10\lambda^2 - 28\lambda + 24$

The characteristic equation of A is

$|A - \lambda I| = 0$

$\Rightarrow \lambda^3 - 10\lambda^2 + 28\lambda - 24 = 0$

$\Rightarrow (\lambda-2)^2(\lambda-6) = 0$

$\Rightarrow \lambda = 2, 2, 6.$

To find the eigen vectors associated with 2:

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector of A corresponding to the eigen value 2 then $(A-2I)X = 0$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—— (1)}$$

Clearly the coefficient matrix is in echelon form.

$\therefore$ The rank of coefficient matrix = 1

There is $3-1=2$ L.I eigen vectors corresponding to $\lambda = 2$.

from (1), $x_1 + x_2 + x_3 = 0$

Let $x_2 = k_1$, $x_3 = k_2$

where k_1, k_2 are parameters and not both zero simultaneously.

then $x_1 = -(k_1 + k_2)$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 \\ 1k_1 + 0k_2 \\ 0k_1 + 1k_2 \end{bmatrix}$$

$$= k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Here $X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are L.I.

eigen vectors of A corresponding to eigen value $\lambda = 2$

and the set of all eigen vectors of A corresponding to the eigen value 2 is given by $k_1X_1 + k_2X_2$

where k_1, k_2 are parameters and both zero simultaneously.

Similarly we can easily find the eigen vectors corresponding to the eigen value $\lambda = 6$.

H.W. Determine the characteristic roots and the corresponding characteristic vectors of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

* Matrix Polynomial :—

An expression of the form

$F(x) = A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m$,

where $A_0, A_1, A_2, \cdots A_m$ are matrices each of order $n \times n$ over a field F, is called a matrix polynomial of degree m, provided $A_m \neq 0$.

— The symbol x is called indeterminate.

— the matrices themselves are matrix polynomials of zero degree.

— Two matrix polynomials are equal iff the coefficients of like powers of x are equal.

* Addition and Multiplication of Polynomials :—

Let $G(x) = A_0 + A_1 x + A_2 x^2 + \cdots + A_k x^k + \cdots + A_m x^m$ and

$H(x) = B_0 + B_1 x + B_2 x^2 + \cdots + B_m x^m + \cdots + B_k x^k$.

→ we define :— if $m > k$ then

$G(x) + H(x) =$

$(A_0 + B_0) + (A_1 + B_1)x + \cdots + (A_k + B_k)x^k + A_{k+1} x^{k+1} + A_{k+2} x^{k+2} + \cdots + A_m x^m$.

— if $m < k$ then

$G(x) + H(x) = (A_0 + B_0) + (A_1 + B_1)x + (A_2 + B_2)x^2 + \cdots + (A_m + B_m)x^m + B_{m+1} x^{m+1} + \cdots + B_k x^k$.

— if $m = k$ then $G(x) + H(x) = (A_0 + B_0) + (A_1 + B_1)x^1 + (A_2 + B_2)x^2 + \cdots + (A_k + B_k)x^k$.

→ $G(x) \cdot H(x) = A_0 B_0 + (A_0 B_1 + A_1 B_0)x + (A_0 B_2 + A_1 B_1 + A_2 B_0) + \cdots + A_k B_m x^{k+m}$.

Note :- (1). The degree of the product of two matrix polynomials is less than or equal to the sum of their degrees.

Imp

(2) Every square matrix over a field F whose elements are ordinary polynomials in x over F, can essentially be expressed as a matrix polynomial in x of degree m, where m is the index of the highest power of x, occuring in any element of the matrix.

Ex:— Let

$$A = \begin{bmatrix} 1+2x+3x^2 & x^2 & 6-4x \\ 1+x^3 & 3-4x^2 & 1-2x+4x^3 \\ 2-3x+2x^3 & 5 & 6x^2+7x \end{bmatrix}$$

then

$$A = \begin{bmatrix} 1+2x+3x^2+0x^3 & 0+0x+x^2+0x^3 & 6-4x+0x^2+0x^3 \\ 1+0x+0x^2+x^3 & 3+0x-4x^2+0x^3 & 1-2x+0x^2+4x^3 \\ 2-3x+0x^2+2x^3 & 5+0x+0x^2+0x^3 & 0+7x+6x^2+0x^3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 6 \\ 1 & 3 & 1 \\ 2 & 5 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & -4 \\ 0 & 0 & -2 \\ -3 & 0 & 7 \end{bmatrix} x + \begin{bmatrix} 3 & 1 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 6 \end{bmatrix} x^2 + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 4 \\ 2 & 0 & 0 \end{bmatrix} x^3$$

which is a matrix polynomial of degree 3.

* The Cayley-Hamilton Theorem

Statement: Every square matrix satisfies its characteristic equation.

Proof :- Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}, \quad I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$$

then the characteristic matrix of A is

$$A-\lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

the characteristic polynomial of A is $|A-\lambda I|$

$$= \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

$$= (-1)^n[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n] \text{ (say)} \quad \text{--- (1)}$$

where $a_1, a_2, \ldots, a_n \in F$

the characteristic equation of A is

$$|A-\lambda I| = 0$$

i.e. $\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0$

Now we prove that

$$A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_{n-1}A + a_nI = 0$$

Since all the elements of $A-\lambda I$ are at most of first degree in λ, all the elements of adj $(A-\lambda I)$ are polynomials of degree $(n-1)$ (or) less.

($\because$ elements of adj$(A-\lambda I)$ are cofactors of the elements of $(A-\lambda I)$).

$\therefore$ adj $(A-\lambda I)$ can be written as a matrix polynomials in λ of degree $(n-1)$.

Let $\operatorname{adj}(A-\lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots + B_{n-2}\lambda^1 + B_{n-1}$.

where $B_0, B_1, B_2, \ldots B_{n-1}$ are square matrices of order n. --- (2)

Now we have

$$(A-\lambda I)\operatorname{adj}(A-\lambda I) = |A-\lambda I| I \quad (\because A \operatorname{adj} A = |A| I)$$

$$\Rightarrow (A-\lambda I)(B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots + B_{n-2}\lambda + B_{n-1}).$$

$$= (-1)^n\left[\lambda^n + a_1\lambda^{n-1} + \cdots + a_n\right]I$$

(by ① & ②)

Comparing coefficients of like powers of λ, we obtain,

$$-B_0 = (-1)^n I$$

$$AB_0 - B_1 = (-1)^n a_1 I$$

$$AB_1 - B_2 = (-1)^n a_2 I$$

$$\cdots\cdots$$

$$-AB_{n-1} = (-1)^n a_n I.$$

Pre-multiplying the above equations successively by $A^n, A^{n-1}, \ldots I$ and adding, we obtain

$$0 = (-1)^n\left[A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_nI\right]$$

$$\Rightarrow A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_nI = 0 \quad ③$$

$\therefore$ A satisfies the characteristic equation.

Note :- If A is a non-singular matrix then $|A| \neq 0$.

Also $|A| = (-1)^n a_n$

$\therefore a_n \neq 0.$

Now Pre-multiplying ③ by A^{-1}, we get

$$A^{-1}\left[A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_{n-1}A + a_nI\right] = 0$$

$$\Rightarrow A^{n-1} + a_1A^{n-2} + a_2A^{n-3} + \cdots + a_{n-1}I + a_nA^{-1}I = 0$$

$$\Rightarrow a_nA^{-1} = -\left[A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}I\right]$$

$$\Rightarrow \boxed{A^{-1} = \frac{(-1)}{a_n}\left[A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}I\right]}$$

Problems :-

2006, State Cayley-Hamilton theorem and using it, find the inverse of $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

Sol :- Statement : Every square matrix satisfies its characteristic equation.

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The characteristic matrix of A is

$$(A - \lambda I) = \begin{bmatrix} 1-\lambda & 3 \\ 2 & 4-\lambda \end{bmatrix}$$

The characteristic polynomial of A is $|A-\lambda I|$

$$= \begin{vmatrix} 1-\lambda & 3 \\ 2 & 4-\lambda \end{vmatrix}$$

$$= 4 - 5\lambda + \lambda^2 - 6$$

$$= \lambda^2 - 5\lambda - 2$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \lambda^2 - 5\lambda - 2 = 0$$

The given matrix A satisfies the characteristic equation.

$$\therefore A^2 - 5A - 2I = 0 \quad ①$$

Now multiplying ① by A^{-1}, we get

$$A - 5I - 2A^{-1} = 0$$

$$\Rightarrow 2A^{-1} = A - 5I$$

$$\Rightarrow A^{-1} = \tfrac{1}{2}[A - 5I] \quad ②$$

$$A-5I = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 3 \\ 2 & -1 \end{bmatrix}$$

$\therefore$ (2) $\equiv$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -4 & 3 \\ 2 & -1 \end{bmatrix}$$

→ Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$ and verify that it is satisfied by A and hence obtain A^{-1}.

Sol^n :- Given that $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

The characteristic matrix of A is

$$A-\lambda I = \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}$$

The characteristic polynomial of A is $|A-\lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}$

$$= -\lambda^3+6\lambda^2-9\lambda+4$$

$\therefore$ The characteristic equation of A is $|A-\lambda I| = 0$

i.e. $-\lambda^3+6\lambda^2-9\lambda+4 = 0$

$\Rightarrow \lambda^3-6\lambda^2+9\lambda-4 = 0$

By the Cayley-Hamilton theorem

$A^3-6A^2+9A-4I = 0$

Now we verify that

$A^3-6A^2+9A-4I = 0$ —— (1)

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}, \quad A^3 = A^2 \cdot A = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$\therefore$ (1) $\equiv$

$$A^3-6A^2+9A-4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \bar{O}$$

Now multiplying (1) by A^{-1}, we get

$A^2-6A+9I-4A^{-1} = 0$

$\Rightarrow A^{-1} = \frac{1}{4}\left[A^2-6A+9I\right]$ —— (2)

Now $A^2-6A+9I =$

$$\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$\therefore$ from (2),

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

→ Show that the matrix $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ satisfies Cayley-Hamilton theorem. (i.e. verification of C-H theorem.)

→ Find the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify Cayley-Hamilton theorem for this matrix. Find the inverse of the matrix A and also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

2004 → Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 1 & 1 \\ -3 & 3 \end{bmatrix}$. Hence find A^{-1} and A^6.

2002 → Use Cayley-Hamilton theorem, to find the inverse of the following matrix.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

→ If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then show that for every integer $n \geq 3$, $A^n = A^{n-2} + A^2 - I$, hence determine A^{50}.

Sol^n :- If $n=3$ then

$$A^3 = A + A^2 - I \quad \text{--- (1)}$$

Since $A^2 = A \cdot A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Now $A^3 = A^2 \cdot A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Now $A + A^2 - I$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\therefore A^n = A^{n-2} + A^2 - I$ is true for $n = 3$

Suppose $A^n = A^{n-2} + A^2 - I$ is true for $n = k$

$\therefore A^k = A^{k-2} + A^2 - I$.

Now for $n = K+1$;

$$A^{K+1} = A \cdot A^K$$
$$= A \cdot [A^{K-2} + A^2 - I]$$
$$= A^{K-1} + A^3 - A$$
$$= A^{K-1} + A + A^2 - I - A \quad (\text{from } ①)$$
$$= A^{K-1} + A^2 - I$$

$\therefore A^n = A^{n-2} + A^2 - I$ is true for $n = K+1$.

$\therefore$ By mathematical induction, it is true for $n \geq 3$,

$$A^n = A^{n-2} + A^2 - I \quad — ②$$

Now $A^3 = A + A^2 - I$

$$\boxed{A^4 = 2A^2 - I}$$

$$A^6 = A^4 + A^2 - I$$

$$\boxed{A^6 = 3A^2 - 2I}$$

$$A^8 = A^6 + A^2 - I$$

$$\boxed{A^8 = 4A^2 - 3I}$$

$$A^{10} = A^8 + A^2 - I$$

$$\boxed{A^{10} = 5A^2 - 4I}$$

Similarly $A^{12} = 6A^2 - 5I$

$\left(\text{i.e } A^{12} = \frac{12}{2}A^2 - \left(\frac{12}{2} - 1\right)I\right)$

$\vdots$

$$A^{50} = 25A^2 - 24I$$

$\left(\text{i.e. } A^{50} = \frac{50}{2}A^2 - \left(\frac{50}{2} - 1\right)I\right)$

$$\therefore A^{50} = 25\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 24\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

Similarity of Matrices

Defn: Let A and B be two square matrices of order n. Then B is said to be similar to A iff $\exists$ $n\times n$ invertible matrix C such that $AC = CB$. i.e., $B = C^{-1}AC$ (or) $A = CBC^{-1}$.

Ex:- Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Sol^n $|C| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{vmatrix} = 1 \neq 0$

$\therefore$ C is non-singular.

$\therefore$ C^{-1} exists.

$\therefore$ C is an invertible matrix.

Now $C^{-1} = \dfrac{\text{adj } C}{|C|}$

$$= \frac{1}{1}\begin{bmatrix} 16-9 & -12+9 & 9-12 \\ -4+3 & 4-3 & 3-3 \\ 3-4 & -3+3 & 4-3 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Now $B = C^{-1}AC$

$$= \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 17 & 16 \\ 5 & 18 & 16 \\ 5 & 17 & 17 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 14 & 13 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ B is similar to A.

Theorem $\Rightarrow$ "Similarity of matrices" is an equivalence relation in the set of all $n \times n$ matrices.

proof: Let A and B be two square matrices of order n. Then B is similar to A if $\exists$ an $n \times n$ invertible matrix C such that

$$B = C^{-1}AC \quad \text{or} \quad A = CBC^{-1}.$$

(i) **Reflexivity:**

Let A be an $n \times n$ matrix then $\exists$ an $n \times n$ invertible matrix I such that $A = I^{-1}AI$

$\Rightarrow$ A is similar to itself.

$\therefore$ Similarity of matrices is reflexive.

(ii) **Symmetry:**

A is similar to B

$\Rightarrow A = CBC^{-1}$

$\Rightarrow AC = CB$

$\Rightarrow C^{-1}AC = B$

$\Rightarrow (C^{-1})A(C^{-1})^{-1} = B$

$\Rightarrow$ B is similar to A. ($\because C^{-1}$ is invertible)

$\therefore$ Similarity of matrices is symmetric.

(iii) Transitivity:

Let A, B and C be three square matrices of order $n\times n$ such that A is similar to B and B is similar to C.

$\Rightarrow \exists$ invertible $n\times n$ matrices P, Q such that
$A = PBP^{-1}$ and $B = QCQ^{-1}$

$\Rightarrow A = P(QCQ^{-1})P^{-1}$

$\Rightarrow A = (PQ)C(Q^{-1}P^{-1})$

$= (PQ)C(PQ)^{-1}$ ($\because P, Q$ are invertible $\Rightarrow PQ$ is also invertible)

$\Rightarrow$ A is similar to C

$\therefore$ Similarity of matrices is transitive.

$\therefore$ "Similarity of matrices" is an equivalence relation.

Note: [1]. If A is similar to B then B is similar to A and we say that A and B are similar.

[2]. If A and B are similar and B and C are similar then A and C are similar.

Theorem → Similar matrices have the same determinant.

Proof: Let A and B be similar matrices.
Then $\exists$ an invertible matrix P such that
$B = P^{-1}AP$.

$$\Rightarrow |B| = |P^{-1}AP|$$
$$= |P^{-1}||A||P|$$
$$= |P^{-1}|\,|P|\,|A|$$
$$= |P^{-1}P|\,|A|$$
$$= |I||A|$$
$$= |A|$$
$$\therefore |B| = |A|.$$

Theorem 2800 ISM: Similar matrices have the same characteristic polynomial and hence the same characteristic roots.

Proof: Let A and B be similar matrices.

Then $\exists$ an invertible matrix P such that

$$B = P^{-1}AP$$
$$\therefore B - \lambda I = P^{-1}AP - \lambda I$$
$$= P^{-1}AP - \lambda P^{-1}P$$
$$= P^{-1}AP - P^{-1}\lambda P$$
$$= P^{-1}AP - P^{-1}\lambda(IP)$$
$$= P^{-1}AP - P^{-1}(\lambda I)P$$
$$= P^{-1}(A - \lambda I)P$$
$$\therefore |B - \lambda I| = |P^{-1}(A - \lambda I)P|$$
$$= |P^{-1}|\,|A - \lambda I|\,|P|$$
$$= |A - \lambda I|\,|P^{-1}P|$$
$$= |A - \lambda I|$$
$$\therefore |B - \lambda I| = |A - \lambda I|$$

$\therefore$ A and B have the same characteristic polynomial and hence same characteristic roots.

Ex: The similar matrices $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 14 & 13 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ have the same characteristic roots.

Soln: The characteristic equ. of A is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 1, 1, 5.$$

Now the characteristic equation of B is

$$|B-\lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 5-\lambda & 14 & 13 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 1, 1, 5.$$

$\therefore$ The similar matrices A & B have the same characteristic roots.

Note: If two matrices (of same order) have same characteristic roots then it is not necessary that they are similar.

Ex: $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & -1 \\ -3 & -2 & 3 \end{bmatrix}$

have the same characteristic roots but are not similar.

Theorem: If x is a characteristic vector of A corresponding to the characteristic root λ then $P^{-1}x$ is a characteristic vector of B corresponding to characteristic root λ where $B = P^{-1}AP$.

proof: ~~If λ is a characteristic root of A and x is~~ corresponding characteristic vector then

$$Ax = \lambda x \quad \longleftarrow (1)$$

Since $B = P^{-1}AP$

$$\Rightarrow B(P^{-1}x) = (P^{-1}AP)(P^{-1}x)$$
$$= P^{-1}A(PP^{-1})x$$
$$= P^{-1}AIx$$
$$= P^{-1}Ax$$
$$= P^{-1}\lambda x \quad (\because Ax = \lambda x)$$
$$= P^{-1}(x\lambda)$$
$$= (P^{-1}x)\lambda$$
$$= \lambda(P^{-1}x)$$

$\therefore B(P^{-1}x) = \lambda(P^{-1}x)$

$\therefore P^{-1}x$ is a characteristic vector of B corresponding to the characteristic root λ.

Theorem $\rightarrow$ If the matrix A is similar to a diagonal matrix D, then the diagonal elements of D are the characteristic roots of A.

proof: Since A and D are similar.

$\therefore$ They have the same characteristic roots.

But the characteristic roots of diagonal

matrix D are its diagonal elements.

Hence the characteristic roots of A are the diagonal elements of D.

Diagonalizable matrix:

If a matrix A is similar to a diagonal matrix then A is said to be diagonalizable.

i.e., a matrix A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP = D$ where D is diagonal matrix.

Also P is said to diagonalize A or transform A to a diagonal form.

Theorem An n-rowed square matrix is diagonalizable iff the matrix possesses 'n' linearly independent characteristic vectors.

proof: Suppose an n-rowed square matrix A is diagonalizable.

Then ~~A is~~ similar to the diagonal matrix

$$D = dia[\lambda_1, \lambda_2 \cdots \lambda_n]$$

$\therefore \exists$ an $n\times n$ invertible matrix $P = [x_1, x_2 \cdots x_n]$

such that $P^{-1}AP = D$.

$\Rightarrow AP = PD$

$\Rightarrow A[x_1, x_2 \cdots x_n] = [x_1\, x_2 \cdots x_n]\, dia[\lambda_1, \lambda_2 \cdots \lambda_n]$

$\Rightarrow [Ax_1, Ax_2 \cdots Ax_n] = [\lambda_1 x_1\ \lambda_2 x_2 \cdots \lambda_n x_n]$

$\Rightarrow Ax_1 = \lambda_1 x_1;\ Ax_2 = \lambda_2 x_2, \cdots\cdots Ax_n = \lambda x_n$

$\Rightarrow x_1, x_2, \cdots x_n$ are characteristic vectors of A corresponding to $\lambda_1, \lambda_2, \cdots \lambda_n$ respectively.

Since the matrix P is non-singular matrix

$\therefore$ Its coloumn vectors $x_1, x_2, \cdots\cdots x_n$ are LI.

$\therefore$ 'A' possesses 'n' LI characteristic vectors.

Conversely, suppose that A possesses 'n' LI characteristic vectors $x_1, x_2, \cdots x_n$ and let $\lambda_1 \lambda_2, \cdots \lambda_n$ be the corresponding characteristic roots then $Ax_1 = \lambda_1 x_1,\ Ax_2 = \lambda_2 x_2, \cdots Ax_n = \lambda x_n$

Let $P = [x_1, x_2, \cdots x_n]$ and $D = dia[\lambda_1, \lambda_2, \cdots \lambda_n]$

then $AP = A[x_1, x_2, \cdots x_n]$

$= [Ax_1, Ax_2 \cdots Ax_n]$

$= [\lambda_1 x_1, \lambda_2 x_2, \cdots \lambda_n x_n]$

$= [x_1, x_2, \cdots x_n]\ dia[\lambda_1, \lambda_2, \cdots \lambda_n]$

$= PD$

$\therefore AP = PD$

$\Rightarrow P^{-1}AP = P^{-1}PD$

$\Rightarrow P^{-1}AP = D$

$\Rightarrow A$ is similar to a diagonal matrix D.

$\Rightarrow A$ is diagonalizable.

Note: In the proof of the above theorem we shown that if A is diagonalizable and P diagonalizes A, then $P^{-1}AP = D$

$$= \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

iff the j^{th} coloumn of P is an eigen vector of A corresponding to the eigen value λ_j of A. $(j = 1, 2, \ldots n)$. The diagonal elements of D are the eigen values of A and they occur in the same order as is the order of their corresponding eigenvectors in the coloumn vectors of P.

Ex: The matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ has characteristic roots 5, 1, 1 with corresponding characteristic vectors $(1,1,1)$, $(2,-1,0)$, $(1,0,-1)$ respectively.

Sol^n: Taking $P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

we have $P^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$

and $P^{-1}AP = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

$$= \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} 5 & 2 & 1 \\ 5 & -1 & 0 \\ 5 & 0 & -1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 20 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$$

$\therefore$ A is similar to the diagonal matrix

$D = \text{dia}(5, 1, 1)$.

Note: Every square matrix is not similar to a diagonal matrix.

Ex: $A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$

Sol: The characteristic equation of A is

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ 2 & 2-\lambda & -1 \\ 1 & 2 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda = 1, 1, 1.$

$\therefore$ The characteristic roots of A are 1, 1, 1.

The characteristic vector X of A corresponding to characteristic root $\lambda = 1$:-

$(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Clearly the coefficient matrix is in echelon form.

$\therefore \rho(A) = 2$

$\therefore$ there is $3-2=1$ LI eigen vectors.

and $x_1 - x_2 + x_3 = 0$

$3x_2 - 3x_3 = 0$

$\Rightarrow x_1 - x_2 + x_3 = 0$

$x_2 - x_3 = 0$

$\Rightarrow x_1 = 0$

Let $x_3 = k$, k is arbitrary constant

then $x_2 = k$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ k \\ k \end{bmatrix}$$

$$= k\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = kX_1$$

Here $X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is LI vector corresponding to characteristic root 1

$\therefore$ The matrix A has only 1 LI characteristic vector and is consequently not diagonal matrix.

$\therefore$ The square matrix A is not similar to diagonal matrix.

Note: [1]. If the eigen values of an $n\times n$ matrix are all distinct then it is always similar to a diagonal matrix.

[2]. Two $n\times n$ matrices with the same set of 'n' distinct eigen values are similar.

→ Prove that the matrices $\begin{bmatrix} -10 & 6 & 3 \\ -26 & 16 & 8 \\ 16 & -10 & -5 \end{bmatrix}$ and $\begin{bmatrix} 0 & -6 & -16 \\ 0 & 17 & 45 \\ 0 & -6 & -16 \end{bmatrix}$ are similar.

Solⁿ: The characteristic equation of A is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} -10-\lambda & 6 & 3 \\ -26 & 16-\lambda & 8 \\ 16 & -10 & -5-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda = 0, -1, 2$

The characteristic roots of A are 0, −1, 2 and they are distinct.

Now the characteristic equation of B is $|B-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 0-\lambda & -6 & -16 \\ 0 & 17-\lambda & 45 \\ 0 & -6 & -16-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda = 0, -1, 2.$

∴ The characteristic roots of B are distinct

∴ A and B are similar.

→ Show that the matrices $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ -1 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -10 & -38 & -36 \\ 4 & 14 & 13 \\ 0 & 1 & 1 \end{bmatrix}$ are similar if $P^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Solⁿ: If B is similar to A then $P^{-1}AP = B$

$$\text{Now } P^{-1}AP = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -8 & -6 \\ 1 & 2 & 1 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & -38 & -36 \\ 4 & 14 & 13 \\ 0 & 1 & 1 \end{bmatrix}$$

$= B.$

problems :

→ Show that the rank of every matrix similar to A is the same as that of A.

Solⁿ: Let B be a matrix similar to A.

$\therefore \exists$ a non-singular matrix P such that

$$B = P^{-1}AP.$$

W.K.T the rank of a matrix doesnot change on multiplication by a non-singular matrix.

$$\therefore \rho(P^{-1}AP) = \rho(A)$$

i.e, $\rho(B) = \rho(A)$.

→ Let A and B be n-rowed square matrices and A be non-singular. Show that the matrices $A^{-1}B$ and BA^{-1} have the same eigen values.

Solⁿ: we have $A^{-1}(BA^{-1})A = A^{-1}B$.

$\therefore BA^{-1}$ is similar to $A^{-1}B$.

But the similar matrices have the same eigen values.

$\therefore A^{-1}B$ and BA^{-1} have the same eigen values.

→ If U be a unitary matrix such that $U^{\theta}AU = \text{diag}[\lambda_1, \lambda_2, \dots \lambda_n]$. Show that $\lambda_1, \lambda_2, \dots \lambda_n$ are the eigen values of A.

Solⁿ: Let $\text{diag}[\lambda_1, \lambda_2, \dots \lambda_n] = D$

Since U is unitary.

$$UU^{\theta} = I$$

$\therefore U^{\theta} = U^{-1}$

$\Rightarrow U^{\theta} A U = D$

$\Rightarrow U^{-1} A U = D$

$\therefore$ A is similar to the diagonal matrix D.

But the similar matrices have the same eigen values and eigen values of D are its diagonal elements.

$\therefore \lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A.

→ If A and B are non-singular matrices of order n, show that the matrices AB and BA are similar.

Sol^n: Since A is non-singular.

$\therefore A^{-1}$ exists.

we have $A^{-1}(AB)A = BA$

$\therefore$ AB and BA are similar matrices.

→ A and B are two $n \times n$ matrices with the same set of 'n' distinct eigen values. Show that there exist two matrices P and Q (one of them is non singular) such that $A = PQ$, $B = QP$.

Sol^n: Since A and B have the same set of 'n' distinct eigen values.

$\therefore$ They are similar.

$\therefore \exists$ a non-singular matrix P such that

$P^{-1}AP = B$ —— ①

Let $P^{-1}A = Q$ then

① ≡ $QP = B$

and $P^{-1}A = Q$

$\Rightarrow A = PQ$.

→ Prove that A is similar to a diagonal matrix; then A^T is similar to A.

Sol^n: Let A be similar to a diagonal matrix D then $\exists$ a non-singular matrix P such that

$$P^{-1}AP = D$$
$$\Rightarrow A = PDP^{-1}$$
$$\Rightarrow A^T = (PDP^{-1})^T$$
$$= (P^{-1})^T D^T P^T$$
$$= (P^T)^{-1} D^T P^T$$
$$= (P^T)^{-1} D P^T \quad (\because D \text{ is diagonal} \Rightarrow D^T = D)$$
$$\therefore A^T = (P^T)^{-1} D P^T.$$

$\therefore$ A^T is similar to D.

$\Rightarrow$ D is similar to A^T.

Finally A is similar to D and D is similar to A^T.

$\Rightarrow$ A is similar to A^T.

Algebraic and Geometric multiplicity of a Characteristic root:

If λ_1 is a characteristic root of order t of the characteristic equation $|A-\lambda I|=0$, then t is called the algebraic multiplicity of λ_1.

If s is the number of linearly independent characteristic vectors corresponding to the characteristic value λ_1, then s is called the geometric multiplicity of λ_1.

Here the number of linearly independent solutions of $(A-\lambda_1 I)X=0$ will be s and

$\rho(A-\lambda_1 I) = n-s$.

Ex: (1) For the matrix O_n, zero is the characteristic root of algebraic multiplicity 'n'.

(2) For the matrix I_n, unity is the characteristic root of algebraic multiplicity 'n'.

Note: $\boxed{1}$. The geometric multiplicity of a characteristic root cannot exceed its algebraic multiplicity. i.e, $s \le t$.

$\boxed{2}$. A square matrix is similar to a diagonal matrix iff the geometric multiplicity of each of its characteristic roots is equal to its algebraic multiplicity.

Problems:

→ Show that $A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ is similar to a diagonal matrix. Also find the transforming matrix and diagonal matrix.

(OR)

Show that the characteristic vectors of $A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ are linearly independent. Hence find a diagonal matrix similar to A.

Sol: The characteristic equation of A is

$|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda = 1, 5, 2$

$\therefore$ The Characteristic roots of A are 1, 5, 2

Since the eigen values of the matrix A are all distinct.

$\therefore$ A is similar to a diagonal matrix.

Since the algebraic multiplicity of each eigen value of A is 1.

$\therefore$ there will be one and only one linearly independent eigen vector of A corresponding to each eigen value of A.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the characteristic vector corresponding to a characteristic value

$\therefore$ Characteristic vector of A corresponding to the characteristic value '1' is given by

$(A - 1I)X = 0$

$$\Rightarrow \begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 3R_2 + 5R_1$

$R_3 \to 3R_3 + 2R_1$

$$\Rightarrow \begin{bmatrix} 3 & 2 & -2 \\ 0 & 16 & -4 \\ 0 & 16 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\Rightarrow \begin{bmatrix} 3 & 2 & -2 \\ 0 & 16 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Clearly the coefficient matrix is in echelon form.

$\therefore \rho(A) = 2$

$\therefore$ These equations have $3-2=1$ linearly independent solutions.

$\therefore$ we have

$$3x_1+2x_2-2x_3=0$$

$$16x_2-4x_3=0 \Rightarrow 4x_2-x_3=0$$

$$\Rightarrow x_3=4x_2.$$

Take $x_2=1$ then $x_3=4$.

and $3x_1+2(1)-2(4)=0$

$$\Rightarrow 3x_1-6=0$$

$$\Rightarrow x_1=2$$

$\therefore X_1=\begin{bmatrix}2\\1\\4\end{bmatrix}$ is a characteristic vector of A corresponding to characteristic value 1 of A.

Also the characteristic vector X of A corresponding to the characteristic value 2 is given by

$$(A-2I)X=0$$

$$\Rightarrow \begin{bmatrix}2 & 2 & -2\\-5 & 1 & 2\\-2 & 4 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_2 \to 2R_2+5R_1$
$R_3 \to R_3+R_1$

$$\sim \begin{bmatrix}2 & 2 & -2\\0 & 12 & -6\\0 & 6 & -3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_1 \to \frac{1}{2}R_1,\ R_2 \to \frac{1}{6}R_2,$
$R_3 \to \frac{1}{3}R_3$

$$\sim \begin{bmatrix}1 & 1 & -1\\0 & 2 & -1\\0 & 2 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_3 \to R_3+R_2$

$$\sim \begin{bmatrix}1 & 1 & -1\\0 & 2 & -1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Clearly the coefficient is in echelonform.

$\therefore$ The rank of coefficient matrix $=2$.

So the equations have $3-2=1$ LI solution

we have $x_1+x_2-x_3=0$

$2x_2-x_3=0 \Rightarrow x_3=2x_2$.

Take $x_3=2$ then $x_2=1$

and $x_1=1$

$\therefore X_2=\begin{bmatrix}1\\1\\2\end{bmatrix}$ is a characteristic vector of A corresponding to the characteristic root 2 of A.

Again characteristic vector X of A corresponding to the characteristic root 5 is given by

$(A-5I).X=0$

$$\Rightarrow \begin{bmatrix}-1 & 2 & -2\\ -5 & -2 & 2\\ -2 & 4 & -4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

$\Rightarrow X_3=\begin{bmatrix}0\\1\\1\end{bmatrix}$ is a characteristic vector of A corresponding to the characteristic root 5 of A

Let $P=[X_1 \ X_2 \ X_3]$

$$=\begin{bmatrix}2 & 1 & 0\\ 1 & 1 & 1\\ 4 & 2 & 1\end{bmatrix}$$

The coloumns of P are LI vectors of A corresponding to the characteristic roots 1, 2, 5. respectively.

The matrix P will transform A to diagonal form D is given by the relation $P^{-1}AP=D$.

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 5\end{bmatrix}$$

$\therefore$ The transforming matrix $P = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

and

diagonal matrix $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Note: By actual multiplication we can verify

$P^{-1}AP = D.$

H.W. → Show that the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ is

diagonalizable. Also find the transforming matrix and diagonal matrix.

→ Show that the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is

diagonalizable.

Also find the diagonal form and a diagonalizing matrix P.

Sol^n: The characteristic equation of A is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

$C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 4 & 4 \\ -1-\lambda & 3-\lambda & 4 \\ -1-\lambda & 8 & 7-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda) \begin{vmatrix} 1 & 4 & 4 \\ 1 & 3-\lambda & 4 \\ 1 & 8 & 7-\lambda \end{vmatrix} = 0$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\Rightarrow (1+\lambda) \begin{vmatrix} 1 & 4 & 4 \\ 0 & -1-\lambda & 0 \\ 0 & 4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1+\lambda)(1+\lambda)(3-\lambda) = 0$$

$\Rightarrow \lambda = -1, -1, 3$.

$\therefore$ The characteristic roots of A are $-1, -1, 3$

The eigen vectors X of A corresponding to the characteristic root -1 are given by

$$(A-(-1)I)X = 0$$

$$\Rightarrow \begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- The rank of the coefficient matrix $= 1$

$\therefore$ the equations have $3-1 = 2$ LI solutions.

$\therefore$ we have

$$-8x_1 + 4x_2 + 4x_3 = 0$$

$$\Rightarrow -2x_1 + x_2 + x_3 = 0$$

Let $x_2 = k_1$ and $x_3 = k_2$; k_1, k_2 are arbitrary constants

$$\therefore x_1 = \frac{k_1 + k_2}{2}$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{k_1+k_2}{2} \\ k_1 \\ k_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{k_1}{2} + \frac{k_2}{2} \\ k_1 \\ k_2 \end{bmatrix}$$

$$= k_1\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$$

$$X = k_1 X_1 + k_2 X_2$$

Here $X_1 = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$ & $X_2 = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$ are LI vectors of A corresponding to characteristic root -1.

$\therefore$ The geometric multiplicity of eigen value is equal to its algebraic multiplicity.

Now the eigen vectors X of A corresponding to the eigen value 3 are given by $(A-3I)X=0$

$$\Rightarrow \begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \rightarrow R_1 - R_3$

$$\sim \begin{bmatrix} 4 & -4 & 0 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 2R_1$

$$\sim \begin{bmatrix} 4 & -4 & 0 \\ 0 & -8 & 4 \\ 0 & -8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \rightarrow R_3 + 4R_1$$

$R_3 \rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 4 & -4 & 0 \\ 0 & -8 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The rank of the coefficient matrix $= 2$

$\therefore$ The equations have $3-2=1$ LI solution

$\therefore$ we have $4x_1 - 4x_2 = 0$

$-8x_2 + 4x_3 = 0$

$\Rightarrow x_1 = x_2$

and $2x_2 = x_3$.

Take $x_3 = 2$

then $x_2 = 1$

and $x_1 = 1$

$\therefore X_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 3.

$\therefore$ The geometric multiplicity of eigen value 3 is 1 and its algebraic multiplicity is also 1.

Since the geometric multiplicity of each eigen value of A is equal to its algebraic multip

∴ A is similar to diagonal matrix.

∴ A is diagonalizable matrix.

Let $P = [x_1, x_2, x_3]$

$$= \begin{bmatrix} 1/2 & 1/2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The coloumns of P are LI eigen vectors of A corresponding to the eigen values $-1, -1, 3$ respectively.

The matrix P will transform A to diagonal form D is given by the relation $P^{-1}AP = D$.

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The transforming matrix $P = \begin{bmatrix} 1/2 & 1/2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

and diagonal matrix $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

H.W. → Show that the matrix $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$ is similar to a diagonal matrix.

Also find the transforming matrix and diagonal matrix.

→ Show that the following matrices are not similar to diagonal matrices:

(i) $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$

Sol^n: (i) $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 3 & 4 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)(2-\lambda)(1-\lambda) = 0.$

$\Rightarrow \lambda = 2, 2, 1.$

$\therefore$ The characteristic roots of A are 2, 2, 1.

The eigen vectors x of A corresponding to the eigen value 2 are given by $(A-2I)X = 0$

$$\Rightarrow \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

The rank of coefficient matrix = 2.

$\therefore$ The equations have $3-2 = 1$ LI solution corresponding to eigen value 2.

The geometric multiplicity of 2 is 1. while its algebraic multiplicity is 2.

Since the geometric multiplicity of this eigen value is not equal to its algebraic multiplicity.

$\therefore$ A is not similar to a diagonal matrix.

Orthogonal vectors:

Inner product space:

Let $V(F)$ be a vector space, where F is the field of real numbers or the field of complex numbers. An inner product on V is a function $f: V \times V \to F$ such that

(i) $f(\alpha, \beta) = \overline{f(\beta, \alpha)}$ where $\overline{f(\beta,\alpha)}$ is the conjugate of the complex number $f(\beta,\alpha)$.

(ii) $f(\alpha,\alpha) > 0$ for $\alpha \neq \bar{0}$ and $f(\alpha,\alpha) = 0$ for $\alpha = \bar{0}$

(iii) $f(a\alpha + b\beta, \gamma) = a f(\alpha,\gamma) + b f(\beta,\gamma)$

where $\alpha, \beta, \gamma, \bar{0} \in V$ and $a, b, 0 \in F$.

The vector space $V(F)$, in which an inner product 'f' defined as above, is called inner product space and is denoted by (V, f).

Note: → In practice $f(\alpha, \beta)$ where $\alpha, \beta \in V$ is denoted by (α, β) or $\langle \alpha, \beta \rangle$ or (α/β)

Here after we use (α, β) for $f(\alpha, \beta)$

→ $f(\alpha,\beta) = (\alpha,\beta)$, the above three conditions of the inner product are written as follows.

(i) $(\alpha,\beta) = \overline{(\beta,\alpha)}$

(ii) $(\alpha,\alpha) > 0$ for $\alpha \neq \bar{0}$ and $(\alpha,\alpha) = 0$ for $\alpha = \bar{0}$

(iii) $(a\alpha + b\beta, \gamma) = a(\alpha,\gamma) + b(\beta,\gamma)$.

→ If $V(F)$ is an inner product space and F is the field of real numbers then $V(F)$ is called Euclidean space.

→ If $V(F)$ is an inner product space and F is the field of complex numbers then $V(F)$ is called unitary space.

Some important observations:

(1) For $0 \in F$ and $\bar{0} \in V \Rightarrow (\bar{0}, \bar{0}) = 0$

(2) for $a \in F$ and $\alpha, \gamma \in V$

$$\Rightarrow (a\alpha, \gamma) = a(\alpha, \gamma)$$

(3) for $\alpha, \beta, \gamma \in V \Rightarrow (\alpha+\beta, \gamma) = (\alpha, \gamma) + (\beta, \gamma)$

(4) for $\bar{0}, \beta \in V \Rightarrow (\bar{0}, \beta) = (0\alpha, \beta) = 0(\alpha, \beta) = 0$

problems:

→ If $\alpha = (a_1, a_2, a_3)$, $\beta = (b_1, b_2, b_3)$ are elements of a vector space $\mathbb{R}^3$ then prove that $(\alpha, \beta) = a_1b_1 + a_2b_2 + a_3b_3$ defines an inner product space.

Sol: Let $\alpha = (a_1, a_2, a_3)$, $\beta = (b_1, b_2, b_3)$ and $\gamma = (c_1, c_2, c_3) \in \mathbb{R}^3$ where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$

(i) $(\alpha, \beta) = a_1b_1 + a_2b_2 + a_3b_3$

$= b_1a_1 + b_2a_2 + b_3a_3$

$= (\beta, \alpha) = \overline{(\beta, \alpha)}$

(ii) $(\alpha, \alpha) = a_1a_1 + a_2a_2 + a_3a_3$

$= a_1^2 + a_2^2 + a_3^2$

If $\alpha = (a_1, a_2, a_3) \neq (0, 0, 0)$

then atleast one of a_1, a_2, a_3 is not zero.

$\therefore (\alpha, \alpha) = a_1^2 + a_2^2 + a_3^2 > 0$

If $\alpha = (a_1, a_2, a_3) = (0, 0, 0)$

then $a_1 = a_2 = a_3 = 0$

$\therefore (\alpha, \alpha) = a_1^2 + a_2^2 + a_3^2$

$= 0.$

(iii) for $a, b \in \mathbb{R}$ and $\alpha, \beta, \gamma \in \mathbb{R}^3$.

$\Rightarrow a\alpha + b\beta = a(a_1, a_2, a_3) + b(b_1, b_2, b_3)$

$= (aa_1, aa_2, aa_3) + (bb_1, bb_2, bb_3)$

$= (aa_1 + bb_1, aa_2 + bb_2, aa_3 + bb_3)$

$\therefore (a\alpha + b\beta, \gamma) = (aa_1 + bb_1)c_1 + (aa_2 + bb_2)c_2 + (aa_3 + bb_3)c_3.$

$= aa_1c_1 + bb_1c_1 + aa_2c_2 + bb_2c_2 + aa_3c_3 + bb_3c_3$

$= (aa_1c_1 + aa_2c_2 + aa_3c_3) + (bb_1c_1 + bb_2c_2 + bb_3c_3)$

$= a(a_1c_1 + a_2c_2 + a_3c_3) + b(b_1c_1 + b_2c_2 + b_3c_3)$

$= a(\alpha, \gamma) + b(\beta, \gamma)$

$\therefore$ The product $(\alpha, \beta) = a_1b_1 + a_2b_2 + a_3b_3$ is an inner product on $\mathbb{R}^3$.

$\mathbb{R}^3$ is an inner product space with the above product.

Note [1]. The inner product of α & β is $(\alpha, \beta) = a_1b_1 + a_2b_2 + a_3b_3$ is called the dot product of α & β and is denoted by $\alpha \cdot \beta$. This is called the standard inner product space in

[2]. If $\alpha = (a_1, a_2, \ldots a_n)$, $\beta = (b_1, b_2, \ldots b_n)$ are two vectors of the vector space $V_n(\mathbb{R})$ then $(\alpha, \beta) = a_1b_1 + a_2b_2 + \ldots + a_nb_n$ is called the standard inner product

[3]. If $\alpha = (a_1, a_2, \ldots a_n)$ and $\beta = (b_1, b_2, \ldots b_n)$ are the elements of the vector space $V_n(C)$ where C is the field of complex numbers then $(\alpha, \beta) = a_1\bar{b}_1 + a_2\bar{b}_2 + \ldots + a_n\bar{b}_n$ — defines an inner product on $V_n(C)$.
It is called standard inner product on $V_n(C)$.

→ **Inner product of two vectors:**

Let X and Y be two complex n-vectors such that $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ & $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

Then the inner product of X & Y denoted by (X, Y) is defined as

$$(X, Y) = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n$$
$$= X^{\theta} Y$$
$$= [\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n]_{1\times n} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1}$$
$$= \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n.$$

→ If X and Y are real n-vectors written as coloumn vectors then their inner-product is defined as

$$(X, Y) = X^T Y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

→ If X and Y are complex n-vectors written as row vectors then

$$(X, Y) = X Y^{\theta}$$
$$= x_1 \bar{y}_1 + x_2 \bar{y}_2 + \cdots + x_n \bar{y}_n.$$

Norm or length of a vector:

Let X be a complex n-vector. The norm (or length) of X denoted by $\|X\|$ is defined as the +ve square root of (X, X).

i.e, Norm or length of $X = \|X\|$

$$= \sqrt{(X, X)}$$
$$= \sqrt{X^{\theta} X}$$
$$= \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$$

where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Unit vector:

If $||x|| = 1$ then x is called a unit vector and is said to be normalized.

– A unit vector is sometimes also called a normal vector.

Ex: Normalize $x = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$$||x|| = \sqrt{(x, x)}$$
$$= \sqrt{x^{\theta} x}$$
$$= \sqrt{x^T x}$$
$$= \sqrt{6} \quad \left(\because [2 \;\; 1 \;\; -1] \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = 6\right)$$

$$\hat{x} = \frac{x}{||x||} = \frac{1}{\sqrt{6}} x$$
$$= \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

which is the required unit vector.

Orthogonal vectors: Let x and y be two complex n-vectors then x is said to be orthogonal to y if $(x, y) = 0$

i.e, $x^{\theta} y = 0$

Orthogonal set: A set S of complex n-vectors $x_1, x_2, \ldots x_k$ is said to be an orthogonal set if any two distinct vectors in S are orthog[onal]

Orthonormal set: A set S of complex n-vectors $x_1, x_2, \ldots x_k$ is said to be an orthonormal set if (i) each vector in S is a unit vector.

(ii) any two distinct vectors in S are orthogonal

Orthogonally similar matrices:

Let A and B be square matrices of order 'n'. Then B is said to be orthogonally similar to A if ∃ an orthogonal matrix P such that

$$B = P^{-1}AP.$$

- If A and B are orthogonally similar, then they are similar also.

→ Every real symmetric matrix is orthogonally similar to a diagonal matrix with real elements.

→ A real symmetric matrix of order 'n' has n mutually orthogonal real eigen vectors.

→ Any two eigen vectors corresponding to two distinct eigen values of real symmetric matrix are orthogonal.

Sol^n: Let X_1, X_2 be two eigen vectors corresponding to two distinct eigen values λ_1, λ_2 of real symmetric matrix A.

Then $AX_1 = \lambda_1 X_1$ —(1) & $AX_2 = \lambda_2 X_2$ —(2)

Here λ_1, λ_2 are real and X_1, X_2 are real vectors.

$$\begin{aligned}
\text{Now } \lambda_1 X_2^T X_1 &= X_2^T(\lambda_1 X_1) \\
&= X_2^T(AX_1) \quad (\text{by } 1) \\
&= (X_2^T A)X_1 \\
&= (X_2^T A^T)X_1 \quad (\because A^T = A) \\
&= (AX_2)^T X_1 \\
&= (\lambda_2 X_2)^T X_1 \quad (\text{by } 2) \\
&= \lambda_2 X_2^T X_1
\end{aligned}$$

$$(\lambda_1 - \lambda_2) X_2^T X_1 = 0$$

$\Rightarrow x_2^T x_1 = 0$ ($\because \lambda_1$ & λ_2 are distinct $\Rightarrow \lambda_1 - \lambda_2 \neq 0$)

$\therefore x_1$ & x_2 are orthogonal

→ If λ occurs exactly 'p' times as an eigen-value of a real symmetric matrix A then A has p but not more than p mutually orthogonal real eigen vectors corresponding to λ.

Working rule for orthogonal reduction of a real symmetric matrix:

- Suppose A is a real symmetric matrix.
- First we find the characteristic roots of A.

If λ is characteristic root of A having p as its algebraic multiplicity then we shall able to find an orthonormal set of p characteristic vectors of A corresponding to this characteristic root.

we should repeat this process for each characteristic root of A.

- Since the characteristic vectors corresponding to two distinct characteristic roots of a real symmetric matrix are mutually orthogonal.

Therefore, the n characteristic vectors found in this manner constitute an orthonormal set.

- The matrix P, having as its columns the members of the orthonormal set obtained above, is orthogonal and is such that $P^{-1}AP = D$ (Diagonal matri

$$\Rightarrow \begin{bmatrix} -13 & 2 & 3 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to R_1 + R_2 + 4R_3$

$$\sim \begin{bmatrix} 1 & 16 & -11 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 16 & -11 \\ 0 & -42 & 28 \\ 0 & -42 & 28 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 16 & -11 \\ 0 & -42 & 28 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 16x_2 - 11x_3 = 0$

and $-42x_2 + 28x_3 = 0 \Rightarrow -3x_2 + 2x_3 = 0$

$\Rightarrow 3x_2 = 2x_3$.

Let $x_3 = k$ then $x_2 = \frac{2k}{3}$

$\Rightarrow x_1 + \frac{32k}{3} - 11k = 0$

$x_1 = \left(11 - \frac{32}{3}\right)k$

$\boxed{x_1 = \frac{1}{3}k}$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}k \\ \frac{2}{3}k \\ k \end{bmatrix}$$

$$= k \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix}$$

$$= kX_3.$$

where $X_3 = \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix}$ is LI Characteristic vector corresponding to characteristic root 14.

$\therefore X_1, X_2, X_3$ are required characteristic vectors corresponding to 0, 0 and 14.

Since the characteristic vectors corresponding to the two distinct characteristic roots of the real symmetric matrix are mutually orthogonal

Now let us normalize the vectors X_1, X_2 and X_3

for this:

$\therefore \|X_1\| = \sqrt{5}$

$\|X_2\| = \sqrt{10}$ and $\|X_3\| = \sqrt{\frac{1}{9}+\frac{4}{9}+1}$

$= \sqrt{14/9} = \frac{\sqrt{14}}{3}$

$$\therefore \hat{X}_1 = \frac{X_1}{\sqrt{5}} = \frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\\0\end{bmatrix} = \begin{bmatrix}-2/\sqrt{5}\\1/\sqrt{5}\\0\end{bmatrix}$$

$$\hat{X}_2 = \frac{X_2}{\sqrt{10}} = \frac{1}{\sqrt{10}}\begin{bmatrix}-3\\0\\1\end{bmatrix} = \begin{bmatrix}-3/\sqrt{10}\\0\\1/\sqrt{10}\end{bmatrix}$$

and $\hat{X}_3 = \dfrac{X_3}{\sqrt{14}/3}$

$$= \frac{3}{\sqrt{14}}X_3 = \frac{3}{\sqrt{14}}\begin{bmatrix}1/3\\2/3\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{14}\\2/\sqrt{14}\\3/\sqrt{14}\end{bmatrix}$$

Let $P = \begin{bmatrix}\hat{X}_1 & \hat{X}_2 & \hat{X}_3\end{bmatrix}$

$$= \begin{bmatrix}-2/\sqrt{5} & -3/\sqrt{10} & 1/\sqrt{14}\\ 1/\sqrt{5} & 0 & 2/\sqrt{14}\\ 0 & 1/\sqrt{10} & 3/\sqrt{14}\end{bmatrix}$$

which is the required orthogonal matrix.

$\therefore P^T P = I$

and P is non-singular.

$\therefore P^{-1}$ exists.

$\boxed{P^T = P^{-1}}$

$\Rightarrow AP = PD$

$\Rightarrow A = PDP^{-1}$

$\Rightarrow A = PDP^{T}$.

$$\Rightarrow A = \begin{bmatrix} 2/\sqrt{5} & 2/\sqrt{6} & 1/\sqrt{5} \\ 0 & 1/\sqrt{6} & 0 \\ 1/\sqrt{5} & 1/\sqrt{6} & -2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 5/\sqrt{5} \\ 0 & 0 & 0 \\ 0 & 0 & -10/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 2/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$

which is the required symmetric matrix.

Unitarily Similar matrices:

Defn: Let A and B be square matrices of order 'n'. Then B is said to be unitarily similar to A if $\exists$ a unitary matrix P such that $B = P^{-1}AP$.

— If A and B are unitarily similar, then they are similar also.

→ Every Hermitian matrix is unitarily similar to a diagonal matrix.

→ An $n \times n$ Hermitian matrix H has 'n' mutually orthogonal eigen vectors in the complex vectorspace V_n.

→ Any two eigen vectors corresponding to two distinct eigen values of a Hermitian matrix are orthogonal.

→ If λ occurs exactly p times as an eigen val of Hermitian matrix A then A has p but not more than p mutually eigen vectors corresponding to λ.

Problem:

Determine the diagonal matrix unitarily similar to the Hermitian matrix $A = \begin{bmatrix} 2 & 1-2i \\ 1+2i & -2 \end{bmatrix}$ obtaining also the transformation matrix.

Sol^n: Given Hermitian matrix $A = \begin{bmatrix} 2 & 1-2i \\ 1+2i & -2 \end{bmatrix}$

Now the characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1-2i \\ 1+2i & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 9 = 0$$

$$\Rightarrow \lambda = -3, 3.$$

∴ The characteristic roots of A are $-3, 3$.

The algebraic multiplicity of characteristic root -3 is 1.

∴ There will be one L.I eigen vector.

The characteristic vector X corresponding to this eigen value is $(A - (-3)I)X = 0$.

$$\Rightarrow \begin{bmatrix} 5 & 1-2i \\ 1+2i & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 5x + (1-2i)\,y = 0$ —①

$(1+2i)\,x + y = 0$ —②

$\Rightarrow x = 1-2i;\ y = -5$

$\therefore X_1 = \begin{bmatrix} 1-2i \\ -5 \end{bmatrix}$.

Similarly corresponding to $\lambda = 3$ the eigen vector

$X_2 = \begin{bmatrix} 5 \\ 1+2i \end{bmatrix}$.

$\therefore X_1$ and X_2 are characteristic vectors corresponding to characteristic values $-3, 3$.

Since the characteristic vectors corresponding to the two distinct characteristic roots of the Hermitian matrix are mutually orthogonal.

Now let us normalize the vectors X_1 & X_2:

For this,

$\|X_1\| = \sqrt{|1-2i|^2 + |-5|^2} = \sqrt{5+25} = \sqrt{30}$

and $\|X_2\| = \sqrt{|5|^2 + |1+2i|^2} = \sqrt{25+5} = \sqrt{30}$.

$\therefore \hat{X}_1 = \dfrac{X_1}{\|X_1\|} = \dfrac{1}{\sqrt{30}}\begin{bmatrix} 1-2i \\ -5 \end{bmatrix} = \begin{bmatrix} \frac{1-2i}{\sqrt{30}} \\ -5/\sqrt{30} \end{bmatrix}$

$\hat{X}_2 = \dfrac{X_2}{\|X_2\|} = \dfrac{1}{\sqrt{30}}\begin{bmatrix} 5 \\ 1+2i \end{bmatrix} = \begin{bmatrix} 5/\sqrt{30} \\ \frac{1+2i}{\sqrt{30}} \end{bmatrix}$

Let $P = [\hat{X}_1 \ \hat{X}_2]$

$= \begin{bmatrix} \frac{1-2i}{\sqrt{30}} & 5/\sqrt{30} \\ -5/\sqrt{30} & \frac{1+2i}{\sqrt{30}} \end{bmatrix} = \dfrac{1}{\sqrt{30}}\begin{bmatrix} 1-2i & 5 \\ -5 & 1+2i \end{bmatrix}$

which is the required unitary matrix.

Since P is unitary matrix.

$\therefore P^\theta P = I$.

and P is non-singular.

$P^\theta = P^{-1}$

$P^{-1} = P^\theta = \dfrac{1}{\sqrt{30}}\begin{bmatrix} 1+2i & -5 \\ 5 & 1-2i \end{bmatrix}$

Since every Hermitian matrix is unitarily similar to a diagonal matrix.

$$\therefore P^{-1}AP = \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix} = \text{dia}(-3,3) = D$$

$\therefore$ D is unitarily similar to A.

where P is transforming matrix.

H.W. $\rightarrow$ Find a unitary matrix that will diagonalize the Hermitian matrix $\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$